# On (logicality), invariance, and definability

Intensionality in Mathematics
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# Introduction

# Invariance and Definability

- An automorphism of a structure $M$ mapping $a$ to $b$. Expressing **similarity**.
- A definable set separating $a$ from $c$. Expressing **dissimilarity**.

What is the relationship between

- having no automorphism mapping $a$ to $b$, and
- having a definable set separating $a$ and $b$?

What is the relationship between

- $A \subseteq \Omega$ is definable, and
- $A$ is invariant wrt all automorphisms?

What about the level of **quantifiers**?

# Quantifiers

**Definition (Lindström, 1966; Mostowski, 1957)**

- A **generalized quantifier** $Q$ of type $\langle n_1, \ldots, n_k \rangle$ is a (class) of structures in the language $\{ \mathtt{R}_1, \ldots, \mathtt{R}_k \}$ where $\mathtt{R}_i$ is of arity $n_i$.
- $M \vDash_s Q\bar{x}_1, \ldots, \bar{x}_k(\varphi_1, \ldots, \varphi_k)$ iff $(\mathrm{dom}(M), \varphi_1^{M,s}, \ldots, \varphi_k^{M,s}) \in Q$.

Examples:

- $\exists = \{ (\Omega, A) \mid A \subseteq \Omega, A \neq \emptyset \}$
- $\forall = \{ (\Omega, \Omega) \mid \top \}$
- $Q_0 = \{ (\Omega, A) \mid A \subseteq \Omega, |A| \geq \aleph_0 \}$
- $I = \{ (\Omega, A, B) \mid \ |A| = |B| \}$

A quantifier $Q$ is **definable in the logic $\mathscr{L}$** if there is $\varphi$ of $\mathscr{L}(\mathtt{R}_1, \ldots, \mathtt{R}_k)$, such that

$$(\Omega, R_1, \ldots, R_k) \vDash \varphi \text{ iff } (\Omega, R_1, \ldots, R_k) \in Q.$$

# Logicality

Logic considers the **form** of sentences and arguments. To determine this form we need to know what the **logical constants** are.

(Mautner, 1946; Tarski, 1986)
Logic is the the study of the invariants wrt the most general transformations.

Compare with Klein's Erlangen program for classifying geometries in terms of invariance.

Tarski's thesis
A quantifier on a domain $\Omega$ is logical iff it is invariant wrt all **permutations** of $\Omega$.

Theorem (McGee, 1996; Krasner, 1938)
A quantifier is permutation invariant iff it is $\mathscr{L}_{\infty\infty}$-definable.

# Invariance

- **Klein's Erlangen Program:** Invariance as the defining property for geometries.
- **Tarski's thesis:** Extend to logics; use invariance as defining property for logics and logical operators. (Tarski, 1986)
- **Idea**: Extend the correspondence of invariance and operators to a (antitone) Galois connection: **Inv** maps invariance criteria to sets of operators, and **Aut** maps sets of operators to invariance critera such that

$$q \subseteq \mathrm{Inv}(G) \text{ iff } G \subseteq \mathrm{Aut}(q).$$

- $\mathrm{Inv}(\mathrm{Aut}(q))$ should correspond to definability in $\mathscr{L}(q)$ for some logic $\mathscr{L}$.
- McGee's result can be interpreted as characterizing $\mathrm{Inv}(\mathrm{Aut}(\emptyset))$ as definability in $\mathscr{L}_{\infty\infty}$.

**Folklore:**

$\{ G \subseteq \mathrm{Sym}(\Omega) \} \rightleftharpoons \{ M \text{ set of relations on } \Omega \}$

least closed group

**Krasner's Galois theory:**

$\{ G \subseteq \mathrm{Sym}(\Omega) \} \rightleftharpoons \{ M \text{ set of infinite-ary rel. on } \Omega \}$

least group — definability in $\mathscr{L}_{\infty\infty}$

**With equality:**

$\{ G \subseteq \mathrm{Sym}(\Omega) \} \rightleftharpoons \{ q \text{ set of quantifiers on } \Omega \}$

least group — definability in $\mathscr{L}_{\infty\infty}$

**Without equality:**

$\{ \Pi \text{ set of similarities on } \Omega \} \rightleftharpoons \{ q \text{ set of quantifiers on } \Omega \}$

least full monoid — $\restriction$-definability in $\mathscr{L}^{-}_{\infty\infty}$

# With equality

# A Galois connection

- Fix a domain $\Omega$. Quantifier means local quantifier on $\Omega$.
- $q$ is a set of quantifiers.
- $G$ subgroup of the full symmetric group $\mathrm{Sym}(\Omega)$.

**Definition**

- Let **Aut**$(q)$ be the group of all permutations of $\Omega$ fixing all quantifiers in $q$:
  $\mathrm{Aut}(q) = \{\, g \in \mathrm{Sym}(\Omega) \mid g(Q) = Q \text{ for all } Q \in q \,\}$.
- Let **Inv**$(G)$ be the set of quantifiers fixed by $G$:
  $\mathrm{Inv}(G) = \{\, Q \mid g(Q) = Q \text{ for all } g \in G \,\}$.

**Theorem (Krasner, 1938, 1950), (B/E)**

- $\mathrm{Aut}(\mathrm{Inv}(G)) = G$
- $\mathrm{Inv}(\mathrm{Aut}(q))$ is the set of quantifiers definable in $\mathscr{L}_{\infty\infty}(q)$

# Proof

**Aut(Inv($G$)) = $G$**: Let $\leq$ well-order $\Omega$, and $Q = \{\, g(\leq) \mid g \in G \,\}$ of type $\langle 2 \rangle$. If $h \in \mathrm{Aut}(\mathrm{Inv}(G))$ then $h(\leq) \in Q$ and so there is $g \in G$ such that $h(\leq) = g(\leq)$, implying $h = g$.

**Inv(Aut($q$)) is the set of $Q$s definable in $\mathscr{L}_{\infty\infty}(q)$**: We assume all quantifiers of type $\langle 1 \rangle$ and $\Omega = \omega$.
$Q' \in \mathrm{Inv}(\mathrm{Aut}(q))$ is defined by

$$\forall x_0, x_1, \ldots \Big[ \bigwedge_{i \neq j} x_i{\neq}x_j \wedge \forall y \bigvee_i y{=}x_i \wedge \bigwedge_{Q \in q} \Big( \big( \bigwedge_{A \in Q} Qy \bigvee_{i \in A} y{=}x_i \big) \wedge \big( \bigwedge_{A \notin Q} \neg Qy \bigvee_{i \in A} y{=}x_i \big) \Big) \rightarrow \bigvee_{A \in Q'} \Big( \bigwedge_{i \in A} Px_i \wedge \bigwedge_{i \notin A} \neg Px_i \Big) \Big]$$

**Theorem**

If $\mathrm{Inv}_m(G)$ are all **monadic** quantifiers invariant wrt $G$ then there is a subgroup $G$ such that $\mathrm{Aut}(\mathrm{Inv}_m(G)) \supsetneq G$.

Proof. Let $G$ be the group of **piecewise monotone** permutations on $\omega$: $g \in S_\omega$ is piecewise monotone if there exists partitions $A_1 \cup \ldots \cup A_k = B_1 \cup \ldots \cup B_k = \omega$ such that $g|A_i$ is the unique increasing function $A_i \to B_i$.
$\mathrm{Aut}(\mathrm{Inv}_m(G))$ is closed in the topology generated by

$$U_{\bar{A},\bar{B}} = \{\, h \in \mathrm{Sym}(\omega) \mid h(A_i) = B_i \text{ all } i < k \,\}$$

as basic open sets, where $\bar{A} = A_0, \ldots, A_{k-1}$ and $\bar{B} = B_0, \ldots, B_{k-1}$ are subsets of $\omega$.
The closure of $G$ is $\mathrm{Sym}(\omega)$.

# Without equality

# Similarity relations

- $\pi$ is a **similarity relation** on $\Omega$ if $\mathrm{dom}(\pi) = \mathrm{rng}(\pi) = \Omega$.
- Every onto function is a similarity relation.
- For every similarity $\pi$ there are onto functions $f, g : \Omega \to \Omega'$ such that $\pi = f \circ g^{-1}$.
- $R \; \pi \; S$ if $\forall \bar{a}, \bar{b} \in \Omega$ such that $\bar{a} \; \pi \; \bar{b}$: $\bar{a} \in R$ iff $\bar{b} \in S$.
- $R$ is **invariant** wrt $\pi$ if $R \; \pi \; R$.

**Theorem (Feferman)**
Quantifiers of type $\langle 1, \ldots, 1 \rangle$ are invariant wrt **similarity relations** iff they are definable in $\mathscr{L}^{-}_{\omega\omega}$.

Can we extend the previous Galois connection to an equality-free setting, generalizing Feferman's thm?

# Leibniz equality

$a \sim_q b$ iff

$$\forall \bar{x} \bigwedge_{\varphi \in \mathscr{L}^{-}_{\infty\infty}(q)} \big(\varphi(a, \bar{x}) \leftrightarrow \varphi(b, \bar{x})\big)$$

Let $q = \{ \top, \bot, Q^E \}$, where $Q^E = \{ \{ 0, 2, 4, \dots \} \}$. Clearly $Q^O = \{ \{ 1, 3, 5, \dots \} \}$ is definable in $\mathscr{L}^{-}_{\omega\omega}(q)$ by

$$Q^E x \, \neg Px.$$

However $\{ 1, 3, 5, \dots \}$ is not definable in $\mathscr{L}^{-}_{\infty\infty}(q)$, so the previous strategy of defining $Q^O$ 'from below' will not work.

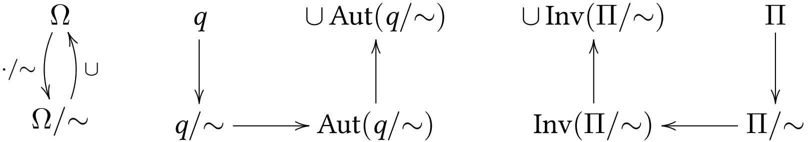
Let $Q^{\restriction q}$ be $Q$ restricted to $\mathscr{L}^{-}_{\infty\infty}(q)$-definable sets.

Observe that

$$\mathscr{L}^{-}_{\infty\infty}(q) \equiv \mathscr{L}^{-}_{\infty\infty}(q^{\restriction q}).$$

# Plan

- Work in $\Omega/\sim_q$ and apply the previous result.




- Observe that $R$ is $\mathscr{L}^{-}_{\infty\infty}(q)$-definable iff it is invariant wrt $\sim_q$. I.e., $\cup(Q/\sim) = Q^{\upharpoonright q}$.
- **Problem:** Can we define $\sim_q$ without knowing the language?
- **Solution:** Yes... sometimes.

# The mappings

- Dually, a set $\Pi$ of similarities defines an equivalence relation by the following condition:

  $a \approx_\Pi b$ if for all $\bar{c} \in \Omega^k$ there is $\pi \in \Pi$ such that $a, \bar{c} \; \pi \; b, \bar{c}$.

**Definition**

A quantifier $Q$ on $\Omega$ is **$\sim$-invariant** wrt $\pi$ if for all relations $R_1, \ldots$, $R_k$, $S_1, \ldots, S_k$ on $\Omega$ **invariant wrt** $\sim$ such that $R_i \; \pi \; S_i$ we have $\langle R_1, \ldots R_k \rangle \in Q$ iff $\langle S_1, \ldots, S_k \rangle \in Q$.

The mappings for the Galois connection can now be defined:

- **Sim**$(q)$ is the set of similarities $\pi$ such that all relations and quantifiers in $q$ are $\sim_q$-invariant wrt $\pi$.
- **Inv**$(\Pi)$ is the set of all relations $R$ and quantifiers $Q$ on $\Omega$ which are $\approx_\Pi$-invariant wrt all similarities in $\Pi$.

# The theorem

- $\Pi$ is a **monoid with involution** if it is closed under composition and taking converses.
- $\Pi$ is **full** if it includes $\approx_\Pi$, is a monoid with involution, and closed under taking subsimilarities, i.e., such that if $\pi \in \Pi$ and $\pi' \subseteq \pi$ is a similarity then $\pi' \in \Pi$.

**Lemma**

- $\sim_q \ = \ \approx_{\mathrm{Sim}(q)}$ and
- If $\Pi$ is full then $\sim_{\mathrm{Inv}(\Pi)} \ = \ \approx_\Pi$.

**Theorem**

Let $q$ be a set of operators then

1. $Q \in \mathrm{Inv}(\mathrm{Sim}(q))$ iff $Q^{\restriction q}$ is definable in $\mathscr{L}^{-}_{\infty\infty}(q)$.
2. $\mathrm{Sim}(\mathrm{Inv}(\Pi))$ is the smallest full monoid including $\Pi$.

# Summary

$$\{\, G \subseteq \mathrm{Sym}(\Omega) \,\} \quad \rightleftharpoons \quad \{\, q \text{ set of quantifiers on } \Omega \,\}$$

least group — definability in $\mathscr{L}_{\infty\infty}$

$$\{\, \Pi \text{ set of similarities on } \Omega \,\} \quad \rightleftharpoons \quad \{\, q \text{ set of quantifiers on } \Omega \,\}$$

least full monoid — $\restriction$-definability in $\mathscr{L}^{-}_{\infty\infty}$

**Open question**

Are all quantifiers invariant wrt all similarities fixing $q$ definable in $\mathscr{L}^{-}_{\infty\infty}(q)$?

# Thank you for your attention.

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