A maximal semantics for dependence logic

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Fredrik Engström

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Team semantics

- **Team semantics:** Lifting semantic values (of formulas) from sets of assignment to sets of sets of assigments (sets of **teams**).
- **Flatness property of FO:** A first-order formula is satisfied by a team iff all assignments satisfy the formula.
- **Subteam property:** If a team satisfies a formula so does each subteam.

The subteam property fails in some logics, e.g., independence logic and exclusion logic.

This talk introduces a logic in which flatness fails.

Dependence Logic

- $\phi ::= \gamma \mid \phi \wedge \phi \mid \phi \vee \phi \mid \exists x \phi \mid \forall x \phi$,
  where γ is a literal or dependence atom.
- $M \models \sigma$ iff $M, \{\emptyset\} \models \sigma$.

- $M, X \models \gamma$ if for all $s \in X$: $M, s \models \gamma$, where γ is a literal.
- $M, X \models =(\bar{t}, t')$ if for all $s, s' \in X$ if $s(\bar{t}) = s'(\bar{t})$ then $s(t') = s'(t')$.
- $M, X \models \phi \wedge \psi$ if $M, X \models \phi$ and $M, X \models \psi$.
- $M, X \models \phi \vee \psi$ if there are $Y \cup Z = X$ such that $M, Y \models \phi$ and $M, Z \models \psi$.
- $M, X \models Qx\phi$ if there is $F : X \to Q_M$ s.t. $M, X[F/x] \models \phi$.

- $\forall_M = \{ M \}$
- $\exists_M = \{ A \subseteq M \mid A \neq \emptyset \}$
- $X[F/x] = \{ s[a/x] \mid s \in X \text{ and } a \in F(s) \}$

Generalized quantifiers

A generalized quantifier Q is a class of structures closed under isomorphisms.

- $Q_M = \{ R \mid (M, R) \in Q \}$.

$$Q_M \subseteq \mathcal{P}(M).$$

$$M, s \vDash Qx\,\phi \text{ iff } \phi^{M,s} \in Q_M$$

- $\forall_M = \{ M \}$
- $\exists_M = \{ A \subseteq M \mid A \neq \emptyset \}$
- $(Q_1)_M = \{ A \subseteq M \mid |A| \geq \aleph_1 \}$

Q is **monotone increasing** if $A \subseteq B$ and $A \in Q_M$ implies $B \in Q_M$.

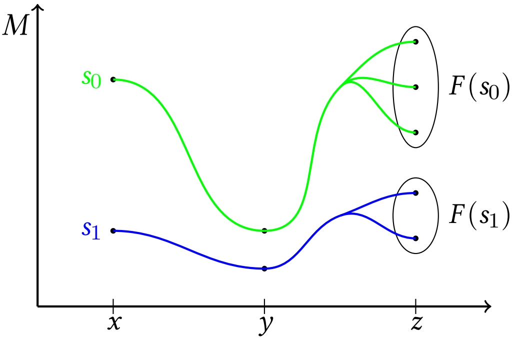
Generalized quantifiers in dependence logic

Works well only for **monotone increasing generalized quantifiers**.

- $M, X \models Qx\,\phi$ iff there is $F : X \to Q_M$ such that $M, X[F/x] \models \phi$.

$$X[F/x] = \{\ s[a/x] \mid s \in X, a \in F(s)\ \}$$

Example: $M, \{s_0, s_1\} \models \exists^{\geq 2} z\, Rxyz$



Iteration and branching

Iteration

$(Q_1 \cdot Q_2)_M = \{\, R \subseteq M^2 \mid \{\, a \mid {}_aR \in (Q_2)_M \,\} \in (Q_1)_M \,\}$

$$(Q_1 \cdot Q_2)xy\,\phi \equiv Q_1 x\, Q_2 y\, \phi$$

For monotone increasing quantifiers:

$\mathrm{Br}(Q_1, Q_2)_M = \{\, R \subseteq M^2 \mid A \times B \subseteq R, A \in (Q_1)_M, B \in (Q_2)_M \,\}$

$$\mathrm{Br}(Q_1, Q_2)xy\,\phi \equiv \begin{pmatrix} Q_1 x \\ Q_2 y \end{pmatrix} \phi$$

Properties of $D(Q)$

- **Empty set property:** $M, \emptyset \models \phi$
- **Downwards closure:** If $Y \subseteq X$ and $M, X \models \phi$ then $M, Y \models \phi$.

Flattness of $FO(Q)$

$$M, X \models \phi \text{ iff for all } s \in X, M, s \models \phi$$

for all FO(Q)-formulas ϕ.

Respect iteration

$$M, X \models (Q_1 \cdot Q_2)xy\phi \text{ iff } M, X \models Q_1 x\, Q_2 x\, \phi$$

Express branching

$$D(Q) \equiv D(Q, \mathrm{Br}(Q, Q))$$

Strength and axiomatizability

Theorem (E., Kontinen)

$$D(Q) \equiv \mathrm{ESO}(Q)$$

Let $\Gamma \vDash_w \phi$ mean that $\Gamma \vDash \phi$ for any monotone increasing interpretation of Q.

Theorem (E., Kontinen, Väänänen)

There is a a sound and complete inference system wrt the following consequence relations:

- $\Gamma \vDash_w \phi$ where ϕ is $\mathrm{FO}(Q, \check{Q})$.
- $\Gamma \vDash \phi$ where ϕ is $\mathrm{FO}(Q_1, \check{Q}_1)$.

Non-monotone quantifiers

$$M \vDash \exists^{=5} x\, Px$$

$$\exists F : \{\, \emptyset \,\} \to \exists_M^{=5},\ \text{s.t. } M, \{\, \emptyset \,\}[F/x] \vDash Px$$

$$\exists A \subseteq M,\ \text{s.t. } |A| = 5 \text{ and } A \subseteq P^M$$

$$M \vDash \exists^{\geq 5} x\, Px$$

ϕ is satisfied by X if

- every assignment $s \in X$ satisfies ϕ.
- ~~every assignment $s \in X$ satisfies ϕ.~~
- for every assignment $s : \mathrm{dom}(X) \to M^k$, $s \in X$ **iff** s satisfies ϕ.

$M, X \vDash_m \phi$ iff $X = \phi^M$ (for first-order ϕ).

Maximal semantics

- $M, X \vDash_m \psi$ if $M, X \vDash \psi$ and for all $Y \supsetneq X : M, Y \nvDash \psi$, for literals ψ.
- $M, X \vDash_m \phi \wedge \psi$ if $\exists Y, Z$ s.t. $X = Y \cap Z$, and both $M, Y \vDash \phi$ and $M, Z \vDash \psi$
- $M, X \vDash_m \phi \vee \psi$ if $\exists Y, Z$ s.t. $X = Y \cup Z$, and both $M, Y \vDash \phi$ and $M, Z \vDash \psi$
- $M, X \vDash_m Qx\phi$ if $\exists Y$ s.t. $Qx\, Y = X$ and $M, Y \vDash \phi$

$$Qx\, X = \{\, s : \mathrm{dom}(X) \setminus \{\, x \,\} \to M \mid \{\, a \in M \mid s[a/x] \in X \,\} \in Q_M \,\}$$

Properties of maximal semantics

- "Maximal dependence logic", D_m, is contained in ESO.
- If $M, X \vDash_m \phi$ then $M, X \vDash \phi$.
- For each ϕ there is X satisfying ϕ.
- There is a formula θ such that $M, X \vDash_m \theta$ if $|M| \geq 2$ and $\mathrm{dom}(X) = \mathrm{FV}(\theta)$.
- There is a translation $^+ : \phi \mapsto \phi^+$ such that

$$M, X \vDash \phi \text{ iff } M, X \vDash_m \phi^+,$$

  for all teams X with $\mathrm{dom}(X) = \mathrm{FV}(\phi)$.
- Thus, $D \leq D_m$ (and $D \equiv D_m$).
- The independence atom is definable in D_m.

In fact, by adding a "universally true" atom to FO (and using maximal semantics) both the dependence and the independence atom are definable.

$D_m(Q)$

Conservativity over $FO(Q)$

$$M, X \vDash_m \phi \text{ iff } X = \phi^M$$

for all FO(Q)-formulas ϕ.

Respect iteration

$$M, X \vDash_m (Q_1 \cdot Q_2)xy\phi \text{ iff } M, X \vDash_m Q_1 x Q_2 x \phi$$

Strength

$$D_m(Q) \equiv D(Q) \equiv ESO(Q)$$

for monotone increasing Q.

Conclusion

- Extending dependence logic with **montone increasing** generalized quantifiers is a natural and stable extension.
- There is a way to introduce non-monotone quantifiers by altering the basic semantics.

Open questions

- $D_m(Q) \equiv D_m(Q, \mathrm{Br}(Q, Q))$? (whenever $\mathrm{Br}(Q, Q)$ makes sense)
- $D_m(Q) \equiv ESO(Q)$?

That's all folks!

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