Models of arithmetic, standardness and omitting types

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Introduction

Map of the talk

Take home message

There are natural notions of saturation (for ctble models of PA) stronger than recursive saturation.

Map of the talk

Take home message
There are natural notions of saturation (for ctble models of PA) stronger than recursive saturation.

- Present the basic background concepts.
- By presenting two examples (maximal automorphisms and the standard predicate) introduce the main concept of the talk, **transplendence**.
- Give definitions and equivalents of **subtransplendence** and **transplendence**.
- Present similar concepts closely related to the **standard predicate**.
- If time permits show an application to satisfaction classes.

Preliminaries

- All languages will be **recursive**, so will all extensions of languages.
- All models will be (mostly non-standard) **models of PA** (even though many results hold in full generality).

Recursive saturation

- A (consistent) type $p(\bar{x}, \bar{a})$ over M is a set of formulas whose free variables are among the $\bar{x}$, and with parameters from M among the $\bar{a}$, (such that $\mathrm{Th}(M, \bar{a}) + p(\bar{x}, \bar{a})$ is consistent).

Definition

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- Any structure has a recursively saturated elementary extension of the same cardinality.

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Definition

A structure is **recursively saturated** if all recursive types are realized.

- Any structure has a recursively saturated elementary extension of the same cardinality.
- The ctble rec sat models are characterizable in terms of their theory and their **standard system**.

Definition

The **standard system** of a model M of PA, $\mathrm{SSy}(M)$, is the set

$$\{ A \cap \mathbb{N} \mid A \in \mathrm{Def}(M) \} .$$

- $\mathrm{Def}(M)$ is the set of all sets definable **with parameters**.

The Standard System

Definition

The **standard system** of a model M of PA, $\mathrm{SSy}(M)$, is the set

$$\{\, A \cap \mathbb{N} \mid A \in \mathrm{Def}(M) \,\}\,.$$

- If M is recursively saturated then a complete consistent type p is realized in M iff $p \in \mathrm{SSy}(M)$.
- A **Scott set** is an ω-model of WKL_0, i.e., a subset of $\mathscr{P}(\mathbb{N})$ closed under unions, relative recursion and weak König's Lemma.
- All standard systems are Scott sets.
- All Scott sets of at most cardinality $\aleph_1$ is the standard system of $M \models \mathrm{PA}$.

Resplendence

- If M is ctble. rec. sat. and T is a theory in a larger language $L' \supset L \cup \{ \bar{a} \}$, $\bar{a} \in M$, consistent with $\mathrm{Th}(M, \bar{a})$ then there is an expansion of $(M, \bar{a})$ satisfying T.
- M is called **resplendent** if such expansions exists for every consistent T.
- Resplendence implies rec. sat.
- On unctble structures resplendence is strictly stronger than rec. sat.

Stronger than resplendence?

- Recursive saturation is enough to give us a rich automorphism group.
- An automorphism is **maximal** if it moves all non-def pts:

$$g \text{ is an automorphism } + \neg\exists x\Big(gx = x \wedge \bigwedge(\exists! y\varphi \to \neg\varphi(x))\Big)$$

- An expansion satisfying a theory **and omitting a type**.
- For countable rec.sat. models existence of maximal automorphism is equivalent to **arithmetical saturation**. (rec.sat. + $(\mathbb{N}, \mathrm{SSy}(M)) \models \mathrm{ACA}_0$) (Kaye et al., 1991b)

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- **The standard predicate** has a similar definition:

$$T_{K=\mathbb{N}} = \{\, K(n) \mid n \in \mathbb{N} \,\} + \neg\exists x\Big(K(x) \wedge \bigwedge_{n\in\mathbb{N}} x > n\Big)$$

- All models have (unique) expansions satisfying that definition. But we might add statements about standard numbers to the theory.

Transplendence

Introduction

- Is there a resplendence like property for $T + p\uparrow$, where $p\uparrow$ expresses that the type p is omitted? (T and p are in extended languages.)
- Clearly $p\uparrow$ might imply that some type in the base language is omitted:

  Let M be any non-prime model of PA, and a non-definable. Then $\mathrm{Th}(M) + \mathrm{tp}(a)\uparrow$ is consistent.

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  Let M be any non-prime model of PA, and a non-definable. Then $\mathrm{Th}(M) + \mathrm{tp}(a)\uparrow$ is consistent.
- Two ways out: (1) Restrict to types p where $p\uparrow$ doesn't imply any type in the base language is omitted.
- (2) Work in elementary substructures of the base structure.

Subtransplendence

- Two ways out: (1) Restrict to types p where $p\!\uparrow$ doesn't imply any type in the base language is omitted.
- **(2) Work in elementary substructures of the base structure.**

Definition
M is **subtransplendent** if for all $T, p(\bar{x}) \in \mathrm{SSy}(M)$ in $\mathscr{L} \supseteq \mathscr{L}_0 \cup \{\bar{a}\}$ such that there is a model of $T + p\!\uparrow + \mathrm{Th}(M, \bar{a})$ there are an elementary submodel $\bar{a} \in N$ of M and an expansion N^+ of N such that $N^+ \models T + p\!\uparrow$.

- A set $\mathscr{X} \subseteq \mathscr{P}(\mathbb{N})$ is a **β-model** if

$$(\mathbb{N}, \mathscr{X}) \prec_{\Sigma^1_1} (\mathbb{N}, \mathscr{P}(\mathbb{N})).$$

Theorem
A model of PA is subtransplendent iff it is rec.sat and $\mathrm{SSy}(M)$ is a β-model.

Transplendence

- Two ways out: **(1) Restrict to types p where $p\!\uparrow$ doesn't imply any type in the base language is omitted.**
- (2) Work in elementary substructures of the base structure.

Definition

$T + p\!\uparrow$ is **fully consistent over M** iff there are an ω-saturated model N of $\mathrm{Th}(M)$ and an expansion of N satisfying $T + p\!\uparrow$.

Definition

M is **transplendent** if for all $T, p(\bar{x}) \in \mathrm{SSy}(M)$ in $\mathscr{L} \supseteq \mathscr{L}_0 \cup \{\bar{a}\}$ such that $T + p\!\uparrow$ is fully consistent over M there is an expansion M^+ of $(M, \bar{a})$ such that $M^+ \models T + p\!\uparrow$ and $\mathrm{Th}(M^+, \bar{a}) + p\!\uparrow$ is fully consistent over M.

Transplendence II

- M is rec.sat. and ctble. and $\mathrm{SSy}(M)$ is closed under taking fully consistent completions of fully consistent theories, then M is transplendent.
- No **nice** characterization of transplendent models in terms of $\mathrm{SSy}(M)$.

Theorem

If M is transplendent then $\mathrm{SSy}(M)$ is a β_ω-model, i.e.,

$$(\mathbb{N}, \mathrm{SSy}(M)) \prec (\mathbb{N}, \mathscr{P}(\mathbb{N})).$$

Translating second-order into first-order

The K-translate of a second-order arithmetic formula is defined by:

$$\begin{aligned}(t = r)^K &\text{ is } t' = r',\\ (t \in V_i)^K &\text{ is } (v_{2i+1})_{t'} \neq 0,\\ (\Psi_1 \vee \Psi_2)^K &\text{ is } \Psi_1^K \vee \Psi_2^K,\\ (\neg\Psi)^K &\text{ is } \neg\Psi^K,\\ (\exists v_i \Psi)^K &\text{ is } \exists v_{2i}(K(v_{2i}) \wedge \Psi^K), \text{ and}\\ (\exists V_i \Psi)^K &\text{ is } \exists v_{2i+1} \Psi^K,\end{aligned}$$

Lemma

$(M, \mathbb{N}) \models \Theta^K(\bar{n}, \bar{a})$ iff
$(\mathbb{N}, \mathrm{SSy}(M)) \models \Theta(\bar{n}, \mathrm{set}_M(a_0), \ldots, \mathrm{set}_M(a_{k-1}))$.

Outline of a proof

Theorem

If M is transplendent then $\mathrm{SSy}(M)$ is a β_ω-model, i.e.,

$$(\mathbb{N}, \mathrm{SSy}(M)) \prec (\mathbb{N}, \mathscr{P}(\mathbb{N})).$$

- Let $(\mathbb{N}, \mathscr{P}(\mathbb{N})) \models \Psi(\bar{A})$, and $a_i \in M$ code A_i.
- Let N be an ω-saturated model of $\mathrm{Th}(M, \bar{a})$.
- Since $\mathrm{SSy}(N) = \mathscr{P}(\mathbb{N})$, the lemma implies $(N, \mathbb{N}) \models \Psi^K(\bar{a})$.
- Therefore $T_{K=\mathbb{N}} + \Psi^K(\bar{a})$ is fully consistent over M.
- Transplendence of M implies that

$$(M, \mathbb{N}) \models \Psi^K(\bar{a}).$$

- The lemma gives us

$$\mathrm{SSy}(M) \models \Psi(\bar{A}).$$

Even stronger?

- For $A \in \mathrm{SSy}(M)$, let $\mathrm{tp}(A)$ be the complete type of A in $(\mathbb{N}, \mathrm{SSy}(M))$.
- If M is transplendent then $\mathrm{tp}(A) \in \mathrm{SSy}(M)$ for all $A \in \mathrm{SSy}(M)$.

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Definition

$\mathrm{SSy}(M)$ is **completion closed** if for every $T_0, T, p \in \mathrm{SSy}(M)$ s.t. $T + p\!\uparrow$ is fully consistent over T_0 then there is a completion $T_c \in \mathrm{SSy}(M)$ s.t. $T_c + p\!\uparrow$ is fully consistent over T_0.

- If M is a ctble rec. sat. model then M is transplendent iff $\mathrm{SSy}(M)$ is completion closed.
- The relationship between β_ω-models, completion closed sets and sets closed under $A \mapsto \mathrm{tp}(A)$ is not clear.

The standard predicate

$\mathbb{N}$-correctness

Definition
M is **$\mathbb{N}$-correct** if whenever $M \prec {}^*M$ and ${}^*M$ is ω-saturated then $(M, \mathbb{N}) \prec ({}^*M, \mathbb{N})$.

This makes sense since we have the following.

Proposition
If $M \equiv N$, $\mathrm{SSy}(M) = \mathrm{SSy}(N)$ and both are recursively saturated then $(M, \mathbb{N}) \equiv (N, \mathbb{N})$.

Theorem
Any transplendent model of PA is $\mathbb{N}$-correct.

$\mathbb{N}$-correctness and canonical extensions

- Any complete theory T has a canonical extension T^ω to the language with K: T^ω is the theory of (M, ω) where M is an ω-saturated model of T.

Proposition
M is $\mathbb{N}$-correct iff all $\bar{a} \in M$ realizes $\mathrm{tp}^\omega(\bar{a})$ in $(M, \mathbb{N})$.

Conjecture (Kaye)
A ctble model is transplendent iff it is $\mathbb{N}$-correct.

- Would mean that transplendence is implied by "standard rec.sat.", recursive saturation with a standard predicate.
- Compare with the case of rec. sat. and resplendence.

Fullness

- $\mathrm{SSy}(M, \mathbb{N}) = \{\, A \cap \mathbb{N} \mid A \in \mathrm{Def}(M, \mathbb{N}) \,\}$.
- M is **full** if $\mathrm{SSy}(M) = \mathrm{SSy}(M, \mathbb{N})$.
- Any transplendent model is full.

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- M is **full** if $\mathrm{SSy}(M) = \mathrm{SSy}(M, \mathbb{N})$.
- Any transplendent model is full.

Theorem (Kaye)
M is full iff $(\mathbb{N}, \mathrm{SSy}(M)) \models \mathrm{CA}_0$.

- Proved by an translation of first-order into second-order, interpreting $(M, \mathbb{N})$ in $(\mathbb{N}, \mathrm{SSy}(M, \mathbb{N}))$.
- Suprising (at least to me), since it seems that you could say more in $(M, \mathbb{N})$ than in $(\mathbb{N}, \mathrm{SSy}(M, \mathbb{N}))$.
- The translation uses the fact the the complete type $\mathrm{tp}(\bar{a}) \in \mathrm{SSy}(M)$ (for rec.sat. M):
- A formula $\varphi(\bar{a})$ without K is translated into $\varphi(\bar{x}) \in \mathrm{tp}(\bar{a})$.
- $\forall y \varphi(y, \bar{a})$ is translated into $\forall\, \mathrm{tp}(\bar{a}, b) \supseteq \mathrm{tp}(\bar{a}) \big(\varphi^*(b, \bar{a}) \big)$.

Satisfaction classes

- A **satisfaction class** is a predicate S satisfying the Tarski truth conditions, e.g., $S(\varphi \land \psi) \leftrightarrow S(\varphi) \land S(\psi)$, for all (non-standard) formulas in a model M.
- The existence of a satisfaction class is equivalent to recursive saturation (for ctble models).

Satisfaction classes

- A **satisfaction class** is a predicate S satisfying the Tarski truth conditions, e.g., $S(\varphi \wedge \psi) \leftrightarrow S(\varphi) \wedge S(\psi)$, for all (non-standard) formulas in a model M.
- The existence of a satisfaction class is equivalent to recursive saturation (for ctble models).
- Enayat and Visser (2013) gave a new proof, more easily extended.
- For example, it can be shown that there is an ω-saturated model M with a sat.cl. S such that $S(\epsilon_i)$ iff $i \in \mathbb{N}$, where ϵ_0 is $0 = 0$ and ϵ_{i+1} is $\epsilon_i \wedge \epsilon_i$.
- Therefore, any transplendent model has such a satisfaction class.

Thank you

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