Branching quantifiers, compositionally

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Introduction

Branching in natural languages (FO)

Some relative of each villager and some relative of each townsmen hate each other. (Hintikka, 1974)

$$\begin{pmatrix} \forall x \exists y \\ \forall z \exists w \end{pmatrix} \big(V(x) \wedge T(z) \to (R(x,y) \wedge R(z,w) \wedge H(y,w))\big)$$

Stenius and Barwise, among others, argued for its formalization being equivalent to a first-order statement, i.e., no **essential** use of branching.

$$\forall x \forall z \exists y \exists w \big(V(x) \wedge T(z) \to (R(x,y) \wedge R(z,w) \wedge H(y,w))\big)$$

.

Branching in natural languages (FO)

The richer the country, the more powerful one of its officials. (Barwise, 1979)

$$\begin{pmatrix} \forall x \exists y \\ \forall z \exists w \end{pmatrix} \big(\mathrm{CO}(x, y) \wedge \mathrm{CO}(z, w) \wedge \mathrm{RT}(x, z) \to \mathrm{MPT}(y, w)\big)$$

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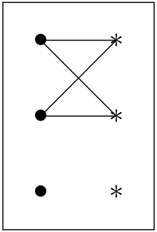
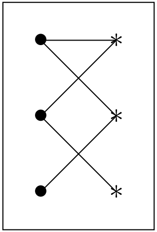
Branching in natural languages (GQ)

Most of the dots and most of the stars are all connected by lines. (Barwise, 1979)



$$\begin{pmatrix} Q_1 x \\ Q_2 y \end{pmatrix} R(x, y)$$



$$Q_1 x\, Q_2 y\, R(x, y)$$
$$Q_2 y\, Q_1 x\, R(x, y)$$

BRANCHING IN NATURAL LANGUAGES (GQ)

Two examiners marked six scripts. (Davies, 1989)

$$\begin{pmatrix} \exists^{\geq 2} x \\ \exists^{\geq 6} y \end{pmatrix} E(x) \wedge S(y) \wedge M(x, y)$$

- Two as $\exists^{\geq 2}$ or $\exists^{=2}$.
- Different readings.

Conclusion

Even if it is discussable if branching of universal and existential quantifiers occur in natural language, it should be clear that branching of **generalized** quantifiers do occur.

.

Branching as an operator

For **monotone** quantifiers the branching of Q_1 and Q_2 as in

$$\begin{pmatrix} Q_1 x \\ Q_2 y \end{pmatrix} R(x, y)$$

can be represented by the quantifier $\mathrm{Br}(Q_1, Q_2)$ as in

$$\mathrm{Br}(Q_1, Q_2)xy\, R(x, y),$$

where

$\mathrm{Br}(Q_1, Q_2)_M$ is

$$\left\{ R \subseteq M^2 \mid \exists A \in Q_{1M}\, \exists B \in Q_{2M} : A \times B \subseteq R \right\}.$$

Compositionality

The meaning of a complex expression is determined by its structure and the meanings of its constituents.

Natural languages are compositional:

- **Productivity**

  The possibility of our understanding sentences which we have never heard before rests evidently on this, that we can construct the sense of a sentence out of parts that correspond to words. (Frege 1914)

- **Systematicity:** There are definite and predictable patterns among the sentences we understand.

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The goal of this talk is to find a compositional analysis of branching quantifiers.

- Let us start with the familiar compositional analysis of branching of the universal and existential quantifiers.

Branching and dependence in logic

- Henkin's partially ordered quantifier prefixes: $\begin{pmatrix} \forall x \exists y \\ \forall z \exists w \end{pmatrix}$ (1959)
- Hintikka and Sandu's IF-logic: $\forall x \exists y \forall z \exists w / x$ (1989)
- Compositionality: Hodges' semantics for IF-logic: Using **sets of assignments**. (1997)
- Vännänen's Dependence Logic: Using dependence as **atomic** property: $\mathrm{D}(x, y, z)$ (2007)

$$\forall x \exists y \forall z \exists w \, (\mathrm{D}(z, w) \wedge \dots)$$

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Hodges' semantics

- X is a **team**, i.e., a set of assignments.
- $M, X \vDash \varphi$.
- For first-order φ: $M, X \vDash \varphi$ iff for all $s \in X$: $M, s \vDash \varphi$.
- $M, X \vDash \mathrm{D}(x, y, z)$ iff there is a function $f : M^2 \to M$ such that for every $s \in X$: $s(z) = f(s(x), s(y))$; or, equivalently:

$$M, X \vDash \mathrm{D}(x, y, z)$$

iff for all $s, s' \in X$ if $s(x) = s'(x)$ and $s(y) = s'(y)$ then $s(z) = s'(z)$.

x	y	z
1	4	4
1	5	4
2	4	2
2	6	2

- $M, X \nvDash x = z$
- $M, X \nvDash x \neq z$
- $M, X \vDash \mathrm{D}(x, z)$
- $M, X \nvDash \mathrm{D}(x, y)$

Hodges' semantics II

- $M, X \vDash \varphi \wedge \psi$ iff $M, X \vDash \varphi$ and $M, X \vDash \psi$.
- $M, X \vDash \varphi \vee \psi$ iff there are Y and Z such that $M, Y \vDash \varphi$ and $M, Z \vDash \psi$ and $X = Y \cup Z$.
- $M, X \vDash \exists x \varphi$ iff there is $f : X \to M$ such that $M, X[f/x] \vDash \varphi$.
- $M, X \vDash \forall x \varphi$ iff $M, X[M/x] \vDash \varphi$.

$$X[f/x] = \{ s[f(s)/x] \mid s \in X \}.$$

$$X[M/x] = \{ s[a/x] \mid a \in M, s \in X \}.$$

- $M \vDash \sigma$ iff $M, \{ \epsilon \} \vDash \sigma$.

Branching in Dependence Logic

$$M \vDash \mathrm{Br}(\forall\exists, \forall\exists)xyzw\, R(x, y, z, w)$$

iff

$$M \vDash \forall x \exists y \forall z \exists w \left(\mathrm{D}(z, w) \wedge R(x, y, z, w)\right)$$

What about generalized quantifiers?

$$M \vDash \mathrm{Br}(Q_1, Q_2)xy\, R(x, y)$$

iff

$$M \vDash Q_1 x\, Q_2 y \left(\mathrm{D}(y) \wedge R(x, y)\right)$$

?

Generalized quantifiers in Dependence Logic

Lifting functions

The **Hodges space** of order ideals on the power set is

$$\mathcal{H}(A) = \mathcal{L}(\mathcal{P}(A)).$$

Given $h : \mathcal{P}(A) \to \mathcal{P}(B)$ we define the **lift**:

$$\mathcal{L}(h) : \mathcal{H}(A) \to \mathcal{H}(B), \mathscr{X} \mapsto \downarrow \{ h(X) \mid X \in \mathscr{X} \},$$

where $\downarrow \mathscr{X}$ is the downward closure of $\mathscr{X}$, i.e.

$$\downarrow \mathscr{X} = \{ X \mid \exists Y \in \mathscr{X}, X \subseteq Y \}.$$

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Lifting quantifers

- Q a monotone type $\langle 1 \rangle$ quantifier.
- $Q_M : \mathcal{P}(M^{n+1}) \to \mathcal{P}(M^n)$
- $\mathcal{L}(Q_M) : \mathcal{H}(M^{n+1}) \to \mathcal{H}(M^n)$
- Gives truth conditions for Q in Hodges semantics:

  $M, X \vDash Qx\,\varphi$ iff there is $F : X \to Q_M$ such that $M, X[F/x] \vDash \varphi$.

  where $X[F/x] = \{ s[a/x] \mid a \in F(s), s \in X \}$.
- $\mathcal{L}$ applied to $\exists$ and $\forall$ give equivalent truth conditions for $\exists$ and $\forall$ as before.

Proposition

For $\mathrm{FO}(Q)$ formulas φ:

$$M, X \vDash \varphi \text{ iff for all } s \in X : \; M, s \vDash \varphi.$$

(Back)slashed generalized quantifiers

Easy to give a definition of slashed and backslashed generalized quantifiers:

Definition

$M, X \models Qx\backslash\bar{y}\varphi$ iff there is $F : X \to Q_M$ such that $M, X[F/x] \models \varphi$ and the value $F(s)$ is determined by $s(\bar{y})$.

Similar for slashed quantifiers $Qx/\bar{y}$.

Proposition

For any X and monotone quantifiers Q_1 and Q_2:

$$M, X \models \mathrm{Br}(Q_1, Q_2)xy\varphi \text{ iff } M, X \models Q_1 x\, Q_2 y/x\, \varphi.$$

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Generalized quantifiers and dependence atoms

If Q_M contains no singletons and $X \neq \emptyset$ then $M, X \not\models Qx\,(\mathrm{D}(x) \wedge \varphi)$.

$$M \models \mathrm{Br}(Q_1, Q_2)xy\,R(x, y)$$

iff

$$M \models Q_1 x\, Q_2 y\,\big(\mathrm{D}(y) \wedge R(x, y)\big)$$

?

Dependence Logic with GQ

Proposition (E, Kontinen)

$D(Q)$ is equivalent to $\mathrm{ESO}(Q)$ on the level of sentences.

- Thus
$$D(\mathrm{Br}(Q_1, Q_2)) \leq D(Q_1, Q_2)$$
and so branching of generalized quantifiers can be defined with the dependence atom.
- Open question: Can this be done compositionally?

Multivalued Dependence

A course database

Course	Student	Credits	Year
LC1510	Svensson	7.5 hp	2010
LC1510	Johansson	7.5 hp	2011
LC1520	Svensson	15 hp	2011
LC1520	AnderssonJohansson	15 hp	2011

- D(Course, Credits)
- It is not true that D(Course, Student).
- F^{Student} takes values for Course and Credits and gives the set of possible values for Student.
- $F^{\text{Student}}(\text{LC1510}, 7.5\text{ hp}) = \{\text{ Svensson}, \text{Johansson }\}$
- F^{Student} is determined by the value of Course.
- $[\text{Course} \twoheadrightarrow \text{Student}]$
- $[\twoheadrightarrow]$ **dependent** on context.
- $F^{\text{Student}}(\text{LC1510}, 7.5\text{ hp}, 2010) = \{\text{ Svensson }\}$
- $F^{\text{Student}}(\text{LC1510}, 7.5\text{ hp}, 2011) = \{\text{ Johansson }\}$
- $[\twoheadrightarrow]$ **not** closed downwards: **Not** true that $[\twoheadrightarrow \text{Student}]$

Multivalued dependence and teams

- If $s \in X$ then $F_X^y(s) = \{ a \mid s[a/y] \in X \}$.

Definition

$M, X \models [\bar{x} \twoheadrightarrow y]$ if F_X^y is determined by the values of $\bar{x}$. (Only for $y \notin \bar{x}$.)

Proposition

$M, X \models [\bar{x} \twoheadrightarrow y]$ iff for all $s, s' \in X$ such that $s(\bar{x}) = s'(\bar{x})$ there exists $s_0 \in X$ such that $s_0(\bar{x}) = s(\bar{x})$, $s_0(y) = s(y)$, and $s_0(\bar{z}) = s'(\bar{z})$, where $\bar{z}$ are the variables in $\mathrm{dom}(X) \setminus (\{ \bar{x} \} \cup \{ y \})$.

- $M, X \models [\bar{x} \twoheadrightarrow y]$ is **dependent on context** and **not closed downwards**.
- $M, X \models \mathrm{D}(\bar{x}, y)$ iff $M, X \models [\bar{x} \twoheadrightarrow y]$ and F_X^y only takes singleton values.

Generalized quantifiers and multivalued dependence

Proposition

If Q_1 and Q_2 are monotone then $M \models \mathrm{Br}(Q_1, Q_2)xy\, R(x, y)$ iff

$$M \models Q_1 x\, Q_2 y \left([\twoheadrightarrow y] \wedge R(x, y) \right).$$

Proposition

FOL + multivalued dependencies has the same strength, on the level of sentences, as ESO, and thus as Dependence Logic.

Proposition [Galliani 2011]

The class of teams definable in FOL + multivalued dependencies are exactly the ones definable in ESO (with an extra predicate for the team).

Embedded multivalued dependence

- Multivalued dependence is **dependent on context**.

Definition

$M, X \models [\bar{x} \twoheadrightarrow \bar{y} \mid \bar{z}]$ iff $Y \models [\bar{x} \twoheadrightarrow \bar{y}]$ where Y is the projection of X onto $\{ \bar{x}, \bar{y}, \bar{z} \}$.

- $[\bar{x} \twoheadrightarrow \bar{y} \mid \bar{z}]$ is **independent on context**.
- This is the **independence atom** introduced by Väänänen and Grädel: $\bar{y} \perp_{\bar{x}} \bar{z}$ iff $[\bar{x} \twoheadrightarrow \bar{y} \mid \bar{z}]$
- However, embedded multivalued dependence is **not** axiomatizable. [Sagiv Walecka 1982] (Which both functional dependence and multivalued dependence are.)
- Embedded multivalued dependence is definable in FOL with multivalued dependencies. [Galliani 2011]

Non monotone quantifiers

Non monotone Quantifiers

Assume Q monotone of type $\langle 1 \rangle$. In the Tarskian setting:

$$M, s \models Qx\varphi \text{ iff } \varphi^s(M) \in Q_M, \text{ or}$$
$$M, s \models Qx\varphi \text{ iff there is } A \subseteq \varphi^s(M) : A \in Q_M.$$

In the Hodges setting this is translated into:

$$M, X \models Qx\varphi \text{ iff there is } F : X \to Q_M \text{ such that } M, X[F/x] \models \varphi.$$

Now assume Q is non monotone, then

$$M, s \models Qx\varphi \text{ iff there is } \textbf{maximal } A \subseteq \varphi^s(M) : A \in Q_M, \text{ or}$$
$$M, s \models Qx\varphi \text{ iff there is } A \subseteq \varphi^s(M) : \text{ if } A \subseteq B \subseteq \varphi^s(M) \text{ then } B \in Q_M.$$

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Non monotone Quantifiers

Definition

Given $F, F' : X \to \mathscr{P}(M)$ let $F \leq F'$ if for every $s \in X$: $F(s) \subseteq F'(s)$.

Let Q be a type $\langle 1 \rangle$ quantifier. Then $M, X \models Qx\varphi$ if there is $F : X \to \mathscr{P}(M)$ such that

1. $M, X[F/x] \models \varphi$ and
2. for each $F' \geq F$ if $M, X[F'/x] \models \varphi$ then for all $s \in X$: $F'(s) \in Q_M$.

Proposition

- For monotone Q the two truth conditions are equivalent.
- For φ in $\mathrm{FO}(Q)$: $M, X \models \varphi$ iff for all $s \in X : M, s \models \varphi$.
- $D(Q)$ is downwards closed.

Sher's maximality principle

Definition (Sher 1990)

A Cartesian product $A \times B$ is **maximal in R** if $A \times B \subseteq R$, no $A' \supsetneq A$ satisfies $A' \times B \subseteq R$ and no $B' \supsetneq B$ satisfies $A \times B' \subseteq R$.

The branching of Q_1 and Q_2, $\mathrm{Br}^S(Q_1, Q_2)$ is

$$\left\{ (M, R) \;\middle|\; R \subseteq M^2, \exists A \in Q_1 \exists B \in Q_2 : A \times B \text{ is maximal in } R \right\}.$$

For monotone Q_1 and Q_2: $\mathrm{Br}^S(Q_1, Q_2) = \mathrm{Br}(Q_1, Q_2)$.

Sher's branching and non monotone GQs

Let Q be a type $\langle 1 \rangle$ quantifier. Then $M, X \vDash Qx \backslash \bar{y}\varphi$ if there is $F : X \to \mathscr{P}(M)$ such that

1. $M, X[F/x] \vDash \varphi$ and
2. for each $F' \geq F$ if $M, X[F'/x] \vDash \varphi$ then for all $s \in X$: $F'(s) \in Q_M$.
3. $F(s)$ is determined by the values $s(\bar{y})$.

Similar for $Qx/\bar{y}$.

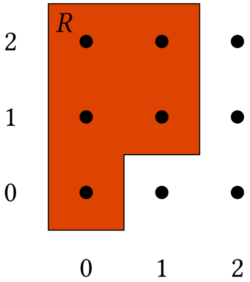
Proposition

Suppose that φ is closed downwards. If $M, X \vDash Q_1 x Q_2 y/x \varphi$ then $M, X \vDash \mathrm{Br}^S(Q_1, Q_2)xy\varphi$.

Sher's branching and non monotone GQs II

Example of a relation R satisfying

- $\mathrm{Br}^S(\exists^{=1}, \exists)xy\,R(x, y)$ and
- $\exists y \exists^{=1} x/y\,R(x, y)$ but
- not $\exists^{=1} x \exists y/x\,R(x, y)$.



This example also shows that in general the two quantifier prefixes $Q_1 x Q_2 y/x$ and $Q_2 y Q_1 x/y$ are not equivalent.

Alternative branching principles

- Barwise (and others): Monotone decreasing quantifiers should be analyzed in the same way as monotone increasing.

Definition

When Q_1 and Q_2 are monotone decreasing

$$\mathrm{Br}(Q_1, Q_2)_M = \left\{ R \subseteq M^2 \;\middle|\; \exists A \in Q_{1M} \exists B \in Q_{2M} : R \subseteq A \times B \right\}.$$

- Westerståhl has generalized this idea to a large family of quantifiers.
- Can any of these principles be analyzed compositionally?

Thank you for your attention.