On dependence and logic

Swedish Congress of Philosophy, Stockholm

Fredrik Engström

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$$\forall x\, \exists y\, Rxy$$

$$\begin{array}{c}\forall x \\ \downarrow \\ \exists y\end{array}$$

$$\forall x \exists y \forall z \exists w Rxyzw$$

$$\begin{array}{c} \forall x \\ \downarrow \\ \exists y \\ \downarrow \\ \forall z \\ \downarrow \\ \exists w \end{array}$$

$$\forall x \exists y \forall z \exists w \, Rxyzw$$

$$\downarrow$$

$$\forall z$$

$$\downarrow$$

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$$
\begin{array}{cc}
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\end{array}
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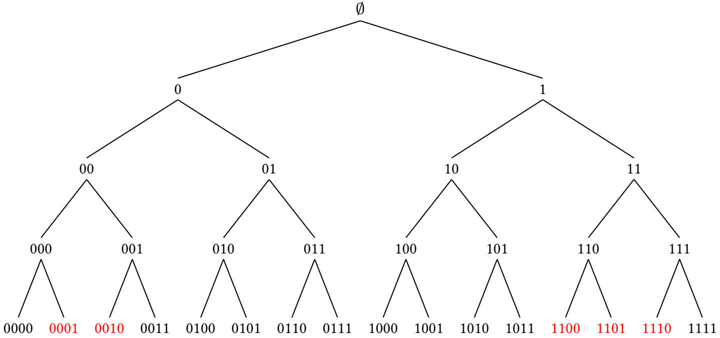

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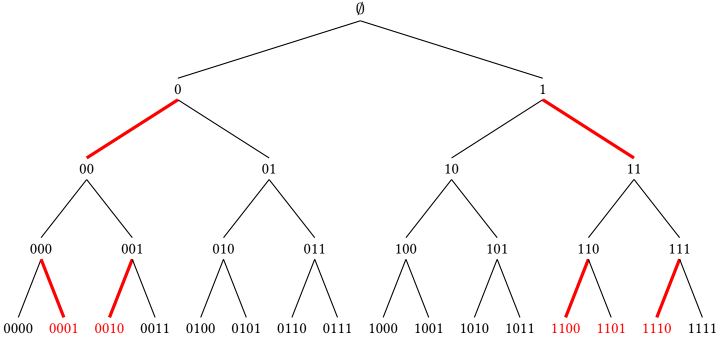

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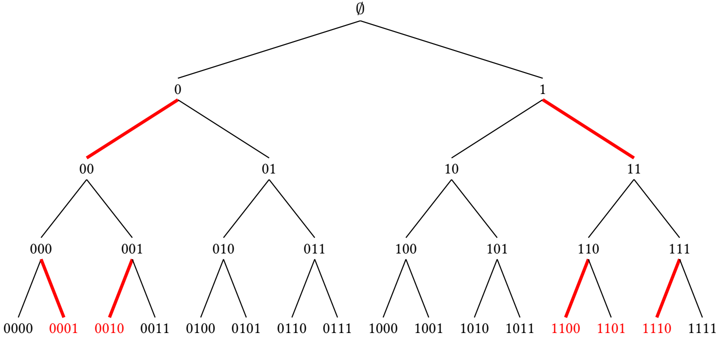

x	y	z	w
0	0	0	1
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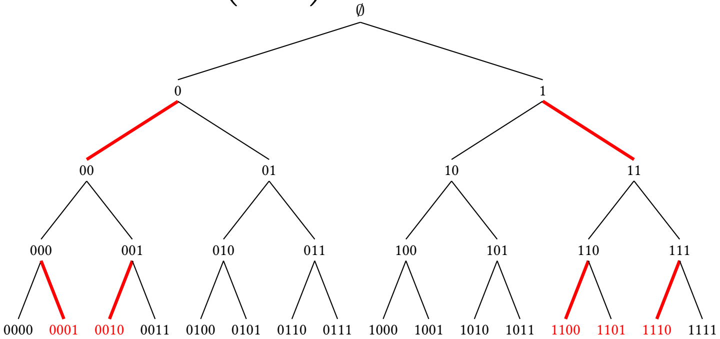

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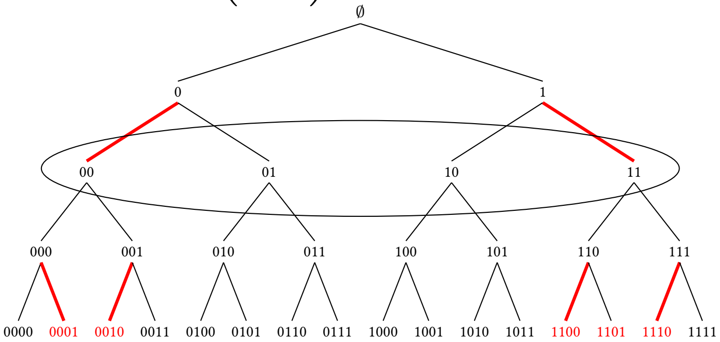

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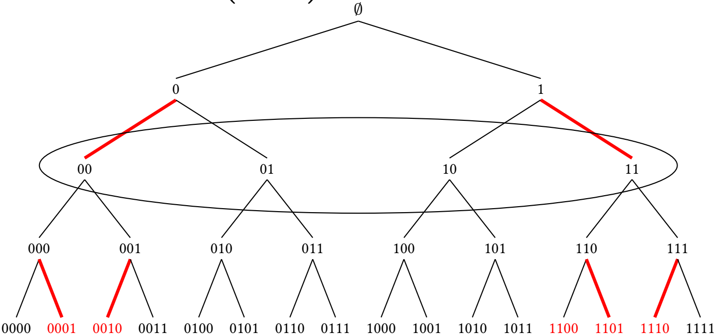

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Dependence

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Take home messages

Dependence is a property of strategies.

strategy $\approx$ set of assignments = **teams**.

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Definition

X a **team**.

$M, X \vDash \mathrm{D}(\bar{x}, y)$ iff for all $s, s' \in X$ if $s(\bar{x}) = s'(\bar{x})$ then $s(y) = s'(y)$.

Dependence logic

- Syntax: FOL + $D(\bar{x}, y)$ in negation normal form
- X is a **team**, i.e., a set of assignments.
- $M, X \models (\neg)R\bar{x}$ iff for all $s \in X$: $M, s \models (\neg)R\bar{x}$.
- $M, X \models \varphi \wedge \psi$ iff $M, X \models \varphi$ and $M, X \models \psi$.

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- $M \models \sigma$ iff $M, \{\emptyset\} \models \sigma$.

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- $M, \emptyset \models \varphi$
- **Downwards closure:** If $Y \subseteq X$ and $M, X \models \varphi$ then $M, Y \models \varphi$.
- Dependence logic $\equiv$ Existential Second Order logic (ESO or Σ^1_1)
- For formulas the situation is slightly different: Dependence logic and the negative fragment of ESO are equivalent.

Dependence logic, again

- $M, X \models \exists x \varphi$ iff there is $f : X \to M$ such that $M, X(x \mapsto f) \models \varphi$.

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Dependence logic, again

- ~~$M, X \models \exists x \varphi$ iff there is $f : X \to M$ such that $M, X(x \mapsto f) \models \varphi$.~~
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The move to set-valued Fs corresponds to **non-deterministic** strategies.

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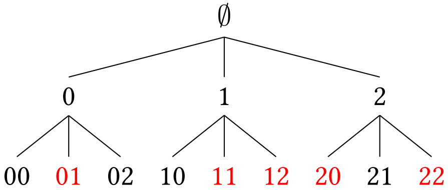
> - **Generalized quantifier**: Q a class of structures (one unary relation).
> - $M, s \vDash Qx \varphi$ iff $(M, \varphi^{M,s}) \in Q$.
> - $Q_M = \{ R \mid (M, R) \in Q \}$.

The move to set-valued Fs corresponds to **non-deterministic** strategies.

Domain $\{0, 1, 2\}$. $\exists^{\geq 2} x \exists^{\geq 2} y Rxy$



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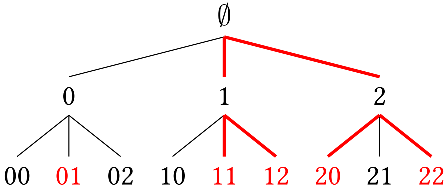

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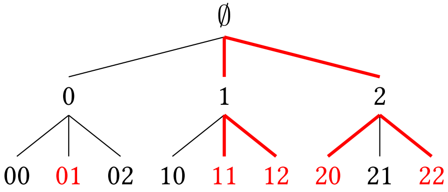

Theorem

For φ in $L(Q)$ (without dependence atoms) and Q **monotone increasing**:

$$M, X \vDash \varphi \text{ iff } M, s \vDash \varphi \text{ for all } s \in X.$$

Example I

$$\exists a \forall x \exists y \forall z \exists w \big(\mathrm{D}(z, w) \wedge (x = z \leftrightarrow y = w) \wedge y \neq a\big)$$

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Example II

$$\begin{aligned}\exists w, w' \Big(\quad & \mathrm{D}(\bar{z}, w) \wedge \mathrm{D}(\bar{z}, w') \wedge \\ & Qx\, \exists y\, (y = w \wedge \mathrm{D}(\bar{z}, x, y) \wedge \\ & Qx'\, \exists y'\, (y' = w' \wedge \mathrm{D}(\bar{z}, x', y') \wedge \\ & \forall u\, \exists v\, (\mathrm{D}(\bar{z}, u, v) \wedge (x = u \to v = w) \wedge \\ & \forall u'\, \exists v'\, (\mathrm{D}(\bar{z}, u', v') \wedge (x' = u' \to v' = w') \wedge \\ & ((v = w \wedge v' = w') \to \varphi(u, u', \bar{z}))))) \Big)\end{aligned}$$

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\begin{pmatrix} Qx \\ Qy \end{pmatrix} \varphi(x, y, \bar{z}) \equiv \mathrm{Br}(Q, Q) xy \varphi(x, y, \bar{z})
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Remember that

$$\begin{pmatrix} \forall x \exists y \\ \forall z \exists w \end{pmatrix} \varphi(x, y, z, w, \bar{z}') \equiv \forall x \exists y \forall z \exists w \left(\mathrm{D}(z, w) \wedge \varphi(x, y, z, w, \bar{z}') \right)$$

Axiomatizations

Dependence: Armstrong's axioms

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If $D \cup \{ \varphi \}$ is a finite set of dependence atoms then $D \vDash \varphi$ iff φ is derivable from D with the following inference rules:

- Reflexivity: If $\bar{y} \subseteq \bar{x}$ then $D(\bar{x}; \bar{y})$.
- Augmentation: If $D(\bar{x}; \bar{y})$ then $D(\bar{x}, \bar{z}; \bar{y}, \bar{z})$.
- Transitivity: If $D(\bar{x}; \bar{y})$ and $D(\bar{y}; \bar{z})$ then $D(\bar{x}; \bar{z})$.

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- The relation $\Gamma \vDash \varphi$ is not r.e.
- Restricting to φ's without dependence atoms.
- An easy exercise shows that $\Gamma \vDash \varphi$ **is** r.e.
- An explicit axiomatization has been given by Kontinen and Väänänen.

Dependence logic with generalized quantifiers

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- An axiomatization of the $L(Q_1)$ consequences of $D(Q_1)$ was recently given.
- Achieved by using a prenex normal form theorem.

Conclusion

Extending dependence logic with generalized quantifiers is a natural and **stable** extension:

- The satisfaction relation is naturally defined when moving to non-deterministic strategies.
- $D(Q)$ properly extends both $L(Q)$ and D.
- $D(Q)$ is in fact equivalent to $\mathrm{ESO}(Q)$.
- $D(Q)$ has a prenex normal form theorem.
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What about the **non-monotone** case?

Thank you for your attention.

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