Logical constants as uniquely definable quantifiers

A proposal by Feferman

Fredrik Engström, Göteborg

September 30, 2011
Background
**GENERATED QUANTIFIERS**

A **generalized quantifier** $Q$ of type $\langle n_1, n_2, \ldots, n_k \rangle$ is a (class) function mapping sets to sets such that

$$Q_M = Q(M) \subseteq \mathcal{P}(M^{n_1}) \times \mathcal{P}(M^{n_2}) \times \cdots \times \mathcal{P}(M^{n_k}).$$

For simplicity we will consider generalized quantifiers of type $\langle 1 \rangle$, i.e, such that $Q_M \subseteq \mathcal{P}(M)$.

**Syntax:** $Qx\varphi$. **Semantics:**

$$M, s \models Qx\varphi \text{ iff } \{ a \in M \mid M, s[a/x] \models \varphi \} \in Q_M$$

- $\forall_M = \{ M \}$
- $\exists_M = \{ A \subseteq M \mid A \neq \emptyset \}$
- $(Q_0)_M = \{ A \subseteq M \mid |A| \geq \aleph_0 \}$
- $(I_a)_M = \{ A \subseteq M \mid a \in A \}$
**Logicality**

Logic considers the **form** of sentences and arguments. To determine this form we need to know what the **logical constants** are.

**Which of the generalized quantifiers should be considered **logical**?**

One answer: The ones that are **topic neutral**. (Ryle, 1954)

- ‘Topic neutral’ as ‘not possible to discriminate between individuals’ gives an **invariance** criterion.
- ‘Topic neutral’ as ‘universally applicable’ gives an **inferential** account.
The semantical viewpoint on logicality

Topic neutrality, and hence logicality, is an invariance criterion.

Invariance under: permutations, bijections, surjections, ….

$Q$ is invariant under bijections if for every bijection $f: M \to N$ we have

$$X \in Q_M \text{ iff } f(X) \in Q_N.$$ 

$I_a$ is not invariant under bijections.

$Q$ is invariant under bijections iff $Q_M(A)$ only depends on the cardinalities of $M$, $A$ and $M \setminus A$.

$$Q_M = \begin{cases} \forall_M & \text{if } |M| = \aleph_{57} \\ \exists_M & \text{otherwise} \end{cases}$$

is invariant under bijections.
**Theorem (McGee 1991 / Krasner 1938)**

$Q$ is bijection invariant iff for each $\kappa$ there is a formula in $L_{\infty \infty}$ defining $Q_\kappa$.

A (global) quantifier $Q$ is invariant under preimages of surjections if for every $h : M \to N$ surjection and for all $A \subseteq N$:

$$h^{-1}(A) \in Q_M \text{ iff } A \in Q_N.$$ 

**Theorem (Feferman)**

Quantifiers of type $\langle 1, \ldots, 1 \rangle$ are invariant under preimages of surjections iff they are definable in $L_{\omega \omega}$.

**Feferman’s Old Thesis**

A quantifier is a logical constant iff it can be defined (in typed $\lambda$-calculus) from equality and monadic quantifiers invariant under preimages of surjections.
THE INFERENTIAL VIEWPOINT

Logicality is the property of being characterizable by inference rules.

**Inferentialism**

*The meaning of logical constants is a matter of the inferential rules govern them.*

Thus, the meaning of conjunction is given by the rules:

\[
\begin{align*}
\frac{\varphi \quad \psi}{\varphi \land \psi} & \quad \frac{\varphi \land \psi}{\varphi} & \quad \frac{\varphi \land \psi}{\psi}
\end{align*}
\]

Prior’s **tonk** was used as an argument against inferentialism:

\[
\begin{align*}
\frac{\varphi \quad \varphi \text{ tonk } \psi}{\varphi \text{ tonk } \psi} & \quad \frac{\varphi \text{ tonk } \psi}{\psi}
\end{align*}
\]

Response: Only some inferential rules do give meanings; when introduction and elimination is in **harmony**.
Harmony

The elimination rules for a certain connective can never allow to deduce more than what follows from the direct grounds of its introduction rules. (Prawitz, 1973)

Harmony could mean:

- Conservativeness (Belnap)
- Normalization / Inversion principle (Prawitz)
- Deductive equilibrium (Tennant)
Uniqueness of definition: Example for $\land$.

Introduce two new symbols $\land_1 \land_2$ with the following rules:

Then $\phi \land_1 \psi \vdash \phi \land_2 \psi$:
A combined approach
Idea: Take the uniqueness criterion seriously. (Example on blackboard.)

Let $L_2$ be pure second order logic:

- Individual variables: $x, y, z, \ldots$,
- Predicate variables (including 0-ary) $P, P_1, \ldots$
- Formulas are built from predicate variables with $\neg, \lor, \land, \rightarrow, \forall, \exists$.

Semantics is Henkin semantics:

- A model $M$ of $L_2$ is a pair of a set $M$ and a set $\text{Pred}(M)$ of subsets of $\mathcal{P}(M^k)$ for the predicate variables to range over.
- Ex: $M$ has two elements 1 and 2, but the unary predicates are only allowed to range over the emptyset and the singleton $\{ 1 \}$. Then
  $$M \models \forall P \forall x(P(x) \rightarrow x = x).$$
Definability

- The language $L_2(Q)$ is $L_2$ extended with a second-order predicate symbol $Q$. Example: $\forall P Q(P)$.
- A model of $L_2(Q)$ gives an interpretation for $Q$ as a second-order predicate, cf. generalized quantifiers.
- We say that a sentence $\sigma$ of $L_2(Q)$ implicitly defines a generalized quantifier $Q$ if for every $L_2$ model $M$ the only second-order predicate satisfying $\sigma$ is $Q_M \cap \text{Pred}(M)$.
- Compare: A formula $\sigma(P)$ of $L_2$ (explicitly) defines a generalized quantifier $Q$ if for every $L_2$ model $M$, for every $R \subseteq M$:

$$(M, R) \models \sigma(P) \iff R \in Q_M.$$
**Logicality**

According to Feferman’s (new) thesis on logicality:

A generalized quantifier $Q$ is **logical** iff it is implicitly definable in $L_2$.

**Main Theorem (Feferman)**

$Q$ is implicitly definable in $L_2$ iff it is (explicitly) definable in FOL.
**Proof of the Main Theorem**

**Beth’s theorem**

Suppose first-order logic. If

\[ T, \sigma(P), \sigma(P') \models \forall \bar{x}(P\bar{x} \leftrightarrow P'\bar{x}) \]

then there is a formula \( \varphi(\bar{x}) \) (without \( P \)) such that

\[ T, \sigma(P) \models \forall \bar{x}(P\bar{x} \leftrightarrow \varphi(\bar{x})). \]

Proof of the Main theorem is by:

- translating to many-sorted first-order logic,
- then using Beth’s theorem for many-sorted formulas (proved by Feferman in 1968) and
- then argue that the many-sorted formula explicitly defining \( Q \) is equivalent to a first-order formula defining \( Q \).
Questions
Why use Henkin semantics for $L_2$?
Locality principle

Whether or not $Q_M(P)$ is true depends only on $M$ and $P$ and not on what sets and relations exist in general over $M$.

This means that we are not forced to consider full second order logic. Not clear what the argument is for considering Henkin (general) semantics for second order logic.
Alternative proof of the main theorem?

Suppose $Q$ of type $\langle 1 \rangle$ is implicitly defined by $\sigma$.
Fix a universe $M$ and for every $A \subseteq M$ let

$$M_A = (M, \{ A \})$$

be the $L_2$ model in which the predicate variables range over the singleton set $\{ A \}$.
$\sigma$ may not include $n$-ary predicate symbols for $n \geq 2$.
Let $Q'_{M} = \mathcal{P}(M)$ be the universally true second order predicate.
Then $(M_A, Q'_M) \models \sigma$ iff $Q'_M \cap \{ A \} = Q_M \cap \{ A \}$ iff $A \in Q_M$.

Let $\varphi$ be the first-order formula we get from $\sigma$ by removing second-order quantifiers and replacing all predicate variables by the single predicate variable $P$. Also replacing all $Q(P)$ by $\top$. Then

$$(M, A) \models \varphi \iff (M_A, Q'_M) \models \sigma \iff A \in Q_M$$

and thus $\varphi$ defines $Q$. 
**Why use** $L_2$?
The black (red?) box

The Main Theorem: plugging in $L_2$ in the machine outputs FOL:

$\text{Beth}^2(L_2, \text{FOL})$

Full second order logic is clearly ‘to strong:’ $Q_0$ is implicitly definable in second order logic.
What about $\Pi_1^1$, or some other fragment more to the nature of deductive rules?
Is this approach really inferential?
- Clearly having a characterization in terms of deduction rules gives implicit definability (in some second order logic).
- What about the other way around?
LUNCH