

# LOGICAL CONSTANTS AS UNIQUELY DEFINABLE QUANTIFIERS

A PROPOSAL BY FEFERMAN

Fredrik Engström, Göteborg

September 30, 2011

# BACKGROUND

## GENERALIZED QUANTIFIERS

A **generalized quantifier**  $Q$  of type  $\langle n_1, n_2, \dots, n_k \rangle$  is a (class) function mapping sets to sets such that

$$Q_M = Q(M) \subseteq \mathcal{P}(M^{n_1}) \times \mathcal{P}(M^{n_2}) \times \dots \times \mathcal{P}(M^{n_k}).$$

For simplicity we will consider generalized quantifiers of type  $\langle 1 \rangle$ , i.e., such that  $Q_M \subseteq \mathcal{P}(M)$ .

Syntax:  $Qx\varphi$ . Semantics:

$$M, s \models Qx\varphi \text{ iff } \{ a \in M \mid M, s[a/x] \models \varphi \} \in Q_M$$

- ▶  $\forall_M = \{ M \}$
- ▶  $\exists_M = \{ A \subseteq M \mid A \neq \emptyset \}$
- ▶  $(Q_0)_M = \{ A \subseteq M \mid |A| \geq \aleph_0 \}$
- ▶  $(I_a)_M = \{ A \subseteq M \mid a \in A \}$

# LOGICALITY

Logic considers the **form** of sentences and arguments. To determine this form we need to know what the **logical constants** are.

Which of the generalized quantifiers should be considered **logical**?

One answer: The ones that are **topic neutral**. (Ryle, 1954)

- ▶ ‘Topic neutral’ as ‘not possible to discriminate between individuals’ gives an **invariance** criterion.
- ▶ ‘Topic neutral’ as ‘universally applicable’ gives an **inferential** account.

## THE SEMANTICAL VIEWPOINT ON LOGICALITY

*Topic neutrality, and hence logicality, is an invariance criterion.*

Invariance under: permutations, bijections, surjections, ...

$Q$  is invariant under bijections if for every bijection  $f: M \rightarrow N$  we have

$$X \in Q_M \text{ iff } f(X) \in Q_N.$$

$I_a$  is not invariant under bijections.

$Q$  is invariant under bijections iff  $Q_M(A)$  only depends on the cardinalities of  $M$ ,  $A$  and  $M \setminus A$ .

$$Q_M = \begin{cases} \forall_M & \text{if } |M| = \aleph_{57} \\ \exists_M & \text{otherwise} \end{cases}$$

is invariant under bijections.

### THEOREM (MCGEE 1991 / KRASNER 1938)

$Q$  is bijection invariant iff for each  $\kappa$  there is a formula in  $L_{\infty\infty}$  defining  $Q_\kappa$ .

A (global) quantifier  $Q$  is invariant under preimages of surjections if for every  $h : M \rightarrow N$  surjection and for all  $A \subseteq N$ :

$$h^{-1}(A) \in Q_M \text{ iff } A \in Q_N.$$

### THEOREM (FEFERMAN)

Quantifiers of type  $\langle 1, \dots, 1 \rangle$  are invariant under preimages of surjections iff they are definable in  $L_{\omega\omega}^-$ .

### FEFERMAN'S OLD THESIS

A quantifier is a logical constant iff it can be defined (in typed  $\lambda$ -calculus) from equality and monadic quantifiers invariant under preimages of surjections.

# THE INFERENCEAL VIEWPOINT

Logicity is the property of being characterizable by inference rules.

## INFERENCEALISM

*The meaning of logical constants is a matter of the inferential rules govern them.*

Thus, the meaning of conjunction is given by the rules:

$$\frac{\varphi \quad \psi}{\varphi \wedge \psi} \quad \frac{\varphi \wedge \psi}{\varphi} \quad \frac{\varphi \wedge \psi}{\psi}$$

Prior's **tonk** was used as an argument against inferentialism:

$$\frac{\varphi}{\varphi \text{ tonk } \psi} \quad \frac{\varphi \text{ tonk } \psi}{\psi}$$

Response: Only some inferential rules do give meanings; when introduction and elimination is in **harmony**.

# HARMONY

*The elimination rules for a certain connective can never allow to deduce more than what follows from the direct grounds of its introduction rules. (Prawitz, 1973)*

Harmony could mean:

- ▶ Conservativeness (Belnap)
- ▶ Normalization / Inversion principle (Prawitz)
- ▶ Deductive equilibrium (Tennant)

# UNIQUENESS

Uniqueness of definition: Example for  $\wedge$ .

Introduce two new symbols  $\wedge_1$   $\wedge_2$  with the following rules

$$\frac{\varphi \quad \psi}{\varphi \wedge_1 \psi} \qquad \frac{\varphi \wedge_1 \psi}{\varphi} \qquad \frac{\varphi \wedge_1 \psi}{\psi}$$

$$\frac{\varphi \quad \psi}{\varphi \wedge_2 \psi} \qquad \frac{\varphi \wedge_2 \psi}{\varphi} \qquad \frac{\varphi \wedge_2 \psi}{\psi}$$

Then  $\varphi \wedge_1 \psi \dashv\vdash \varphi \wedge_2 \psi$ :

$$\frac{\frac{\varphi \wedge_1 \psi}{\varphi} \quad \frac{\varphi \wedge_1 \psi}{\psi}}{\varphi \wedge_2 \psi}$$

# A COMBINED APPROACH

Idea: Take the uniqueness criterion seriously. (Example on blackboard.)

Let  $L_2$  be pure second order logic:

- ▶ Individual variables:  $x, y, z, \dots$ ,
- ▶ Predicate variables (including 0-ary)  $P, P_1, \dots$
- ▶ Formulas are built from predicate variables with  $\neg, \vee, \wedge, \rightarrow, \forall, \exists$ .

Semantics is Henkin semantics:

- ▶ A model  $M$  of  $L_2$  is a pair of a set  $M$  and a set  $\text{Pred}(M)$  of subsets of  $\mathcal{P}(M^k)$  for the predicate variables to range over.
- ▶ Ex:  $M$  has two elements 1 and 2, but the unary predicates are only allowed to range over the emptyset and the singleton  $\{1\}$ .  
Then

$$M \models \forall P \forall x (P(x) \rightarrow x = x).$$

# DEFINABILITY

- ▶ The language  $L_2(\mathbf{Q})$  is  $L_2$  extended with a **second-order** predicate symbol  $\mathbf{Q}$ . Example:  $\forall P\mathbf{Q}(P)$ .
- ▶ A model of  $L_2(\mathbf{Q})$  gives an interpretation for  $\mathbf{Q}$  as a second-order predicate, cf. generalized quantifiers.
- ▶ We say that a sentence  $\sigma$  of  $L_2(\mathbf{Q})$  **implicitly** defines a generalized quantifier  $Q$  if for every  $L_2$  model  $M$  the only second-order predicate satisfying  $\sigma$  is  $Q_M \cap \text{Pred}(M)$ .
- ▶ Compare: A formula  $\sigma(P)$  of  $L_2$  (explicitly) defines a generalized quantifier  $Q$  if for every  $L_2$  model  $M$ , for every  $R \subseteq M$ :

$$(M, R) \models \sigma(P) \text{ iff } R \in Q_M.$$

# LOGICALITY

According to Feferman's (new) thesis on logicality:

A generalized quantifier  $Q$  is **logical** iff  
it is implicitly definable in  $L_2$ .

MAIN THEOREM (FEFERMAN)

$Q$  is implicitly definable in  $L_2$  iff it is (explicitly) definable in FOL.

# PROOF OF THE MAIN THEOREM

## BETH'S THEOREM

Suppose first-order logic. If

$$T, \sigma(P), \sigma(P') \models \forall \bar{x}(P\bar{x} \leftrightarrow P'\bar{x})$$

then there is a formula  $\varphi(\bar{x})$  (without  $P$ ) such that

$$T, \sigma(P) \models \forall \bar{x}(P\bar{x} \leftrightarrow \varphi(\bar{x})).$$

Proof of the Main theorem is by:

- ▶ translating to many-sorted first-order logic,
- ▶ then using Beth's theorem for many-sorted formulas (proved by Feferman in 1968) and
- ▶ then argue that the many-sorted formula explicitly defining  $Q$  is equivalent to a first-order formula defining  $Q$ .

# QUESTIONS

# WHY USE HENKIN SEMANTICS FOR $L_2$ ?

# LOCALITY PRINCIPLE

## LOCALITY PRINCIPLE

*Whether or not  $Q_M(P)$  is true depends only on  $M$  and  $P$  and not on what sets and relations exist in general over  $M$ .*

This means that we are **not forced** to consider full second order logic. Not clear what the argument is for considering Henkin (general) semantics for second order logic.

## ALTERNATIVE PROOF OF THE MAIN THEOREM?

Suppose  $Q$  of type  $\langle 1 \rangle$  is implicitly defined by  $\sigma$ .

Fix a universe  $M$  and for every  $A \subseteq M$  let

$$M_A = (M, \{ A \})$$

be the  $L_2$  model in which the predicate variables range over the singleton set  $\{ A \}$ .

$\sigma$  may not include  $n$ -ary predicate symbols for  $n \geq 2$ .

Let  $Q'_M = \mathcal{P}(M)$  be the universally true second order predicate.

Then  $(M_A, Q'_M) \models \sigma$  iff  $Q'_M \cap \{ A \} = Q_M \cap \{ A \}$  iff  $A \in Q_M$ .

Let  $\varphi$  be the first-order formula we get from  $\sigma$  by removing second-order quantifiers and replacing all predicate variables by the single predicate variable  $P$ . Also replacing all  $Q(P)$  by  $\top$ . Then

$$(M, A) \models \varphi \text{ iff } (M_A, Q'_M) \models \sigma \text{ iff } A \in Q_M$$

and thus  $\varphi$  defines  $Q$ .

# WHY USE $L_2$ ?

# THE BLACK (RED?) BOX

The Main Theorem: plugging in  $L_2$  in the machine outputs FOL:

$$\text{Beth}^2(L_2, \text{FOL})$$

Full second order logic is clearly ‘too strong:’  $Q_0$  is implicitly definable in second order logic.

What about  $\Pi_1^1$ , or some other fragment more to the nature of deductive rules?

IS THIS APPROACH REALLY  
INFERENTIAL?

- ▶ Clearly having a characterization in terms of deduction rules gives implicit definability (in some second order logic).
- ▶ What about the other way around?

# LUNCH