Logical constants as uniquely definable quantifiers

A proposal by Feferman

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Background

Generalized quantifiers

A **generalized quantifier** Q of type $\langle n_1, n_2, \ldots, n_k \rangle$ is a (class) function mapping sets to sets such that

$$Q_M = Q(M) \subseteq \mathcal{P}(M^{n_1}) \times \mathcal{P}(M^{n_2}) \times \ldots \times \mathcal{P}(M^{n_k}).$$

For simplicity we will consider generalized quantifiers of type $\langle 1 \rangle$, i.e, such that $Q_M \subseteq \mathcal{P}(M)$.

Syntax: $Qx\varphi$. Semantics:

$$M, s \vDash Qx\varphi \text{ iff } \{ a \in M \mid M, s[a/x] \vDash \varphi \} \in Q_M$$

- $\forall_M = \{ M \}$
- $\exists_M = \{ A \subseteq M \mid A \neq \emptyset \}$
- $(Q_0)_M = \{ A \subseteq M \mid |A| \geq \aleph_0 \}$
- $(I_a)_M = \{ A \subseteq M \mid a \in A \}$

Logicality

Logic considers the **form** of sentences and arguments. To determine this form we need to know what the **logical constants** are.

Which of the generalized quantifiers should be considered **logical**?

One answer: The ones that are **topic neutral**. (Ryle, 1954)

- ‘Topic neutral’ as ‘not possible to discriminate between individuals’ gives an **invariance** criterion.
- ‘Topic neutral’ as ‘universally applicable’ gives an **inferential** account.

The semantical viewpoint on logicality

Topic neutrality, and hence logicality, is an invariance criterion.

Invariance under: permutations, bijections, surjections, ….
Q is invariant under bijections if for every bijection $f : M \to N$ we have

$$X \in Q_M \text{ iff } f(X) \in Q_N.$$

I_a is not invariant under bijections.
Q is invariant under bijections iff $Q_M(A)$ only depends on the cardinalities of M, A and $M \setminus A$.

$$Q_M = \begin{cases} \forall_M & \text{if } |M| = \aleph_{57} \\ \exists_M & \text{otherwise} \end{cases}$$

is invariant under bijections.

Theorem (McGee 1991 / Krasner 1938)

Q is bijection invariant iff for each κ there is a formula in $L_{\infty\infty}$ defining Q_κ.

A (global) quantifier Q is invariant under preimages of surjections if for every $h : M \to N$ surjection and for all $A \subseteq N$:

$$h^{-1}(A) \in Q_M \text{ iff } A \in Q_N.$$

Theorem (Feferman)

Quantifiers of type $\langle 1, \ldots, 1\rangle$ are invariant under preimages of surjections iff they are definable in $L^{-}_{\omega\omega}$.

Feferman's old thesis

A quantifier is a logical constant iff it can de defined (in typed λ-calculus) from equality and monadic quantifiers invariant under preimages of surjections.

The inferential viewpoint

Logicality is the property of being characterizable by inference rules.

Inferentialism

The meaning of logical constants is a matter of the inferential rules govern them.

Thus, the meaning of conjunction is given by the rules:

$$\frac{\varphi \quad \psi}{\varphi \wedge \psi} \qquad \frac{\varphi \wedge \psi}{\varphi} \qquad \frac{\varphi \wedge \psi}{\psi}$$

Prior's **tonk** was used as an argument against inferentialism:

$$\frac{\varphi}{\varphi \text{ tonk } \psi} \qquad \frac{\varphi \text{ tonk } \psi}{\psi}$$

Response: Only some inferential rules do give meanings; when introduction and elimination is in **harmony**.

Harmony

The elimination rules for a certain connective can never allow to deduce more than what follows from the direct grounds of its introduction rules. (Prawitz, 1973)

Harmony could mean:

- Conservativeness (Belnap)
- Normalization / Inversion principle (Prawitz)
- Deductive equilibrium (Tennant)

Uniqueness

Uniqueness of definition: Example for $\wedge$.
Introduce two new symbols $\wedge_1$ $\wedge_2$ with the following rules

$$\frac{\varphi \quad \psi}{\varphi \wedge_1 \psi} \qquad \frac{\varphi \wedge_1 \psi}{\varphi} \qquad \frac{\varphi \wedge_1 \psi}{\psi}$$

$$\frac{\varphi \quad \psi}{\varphi \wedge_2 \psi} \qquad \frac{\varphi \wedge_2 \psi}{\varphi} \qquad \frac{\varphi \wedge_2 \psi}{\psi}$$

Then $\varphi \wedge_1 \psi \dashv\vdash \varphi \wedge_2 \psi$:

$$\frac{\dfrac{\varphi \wedge_1 \psi}{\varphi} \qquad \dfrac{\varphi \wedge_1 \psi}{\psi}}{\varphi \wedge_2 \psi}$$

A combined approach

Idea: Take the uniqueness criterion seriously. (Example on blackboard.)

Let L_2 be pure second order logic:

- Individual variables: $x, y, z, \ldots,$
- Predicate variables (including 0-ary) $P, P_1, \ldots$
- Formulas are built from predicate variables with $\neg, \vee, \wedge, \rightarrow, \forall, \exists$.

Semantics is Henkin semantics:

- A model M of L_2 is a pair of a set M and a set $\mathrm{Pred}(M)$ of subsets of $\mathcal{P}(M^k)$ for the predicate variables to range over.
- Ex: M has two elements 1 and 2, but the unary predicates are only allowed to range over the emptyset and the singleton $\{ 1 \}$. Then
$$M \models \forall P \forall x(P(x) \rightarrow x = x).$$

Definability

- The language $L_2(\mathbf{Q})$ is L_2 extended with a **second-order** predicate symbol $\mathbf{Q}$. Example: $\forall P \mathbf{Q}(P)$.
- A model of $L_2(\mathbf{Q})$ gives an interpretation for $\mathbf{Q}$ as a second-order predicate, cf. generalized quantifiers.
- We say that a sentence σ of $L_2(\mathbf{Q})$ **implicitly** defines a generalized quantifier Q if for every L_2 model M the only second-order predicate satisfying σ is $Q_M \cap \mathrm{Pred}(M)$.
- Compare: A formula $\sigma(P)$ of L_2 (explicitly) defines a generalized quantifier Q if for every L_2 model M, for every $R \subseteq M$:

$$(M, R) \vDash \sigma(P) \text{ iff } R \in Q_M.$$

Logicality

According to Feferman's (new) thesis on logicality:

A generalized quantifier Q is **logical** iff
it is implicitly definable in L_2.

Main Theorem (Feferman)

Q is implicitly definable in L_2 iff it is (explicitly) definable in FOL.

Proof of the Main Theorem

Beth's theorem

Suppose first-order logic. If

$$T, \sigma(P), \sigma(P') \vDash \forall \bar{x}(P\bar{x} \leftrightarrow P'\bar{x})$$

then there is a formula $\varphi(\bar{x})$ (without P) such that

$$T, \sigma(P) \vDash \forall \bar{x}(P\bar{x} \leftrightarrow \varphi(\bar{x})).$$

Proof of the Main theorem is by:

- translating to many-sorted first-order logic,
- then using Beth's theorem for many-sorted formulas (proved by Feferman in 1968) and
- then argue that the many-sorted formula explicitly defining Q is equivalent to a first-order formula defining Q.

Questions

Why use Henkin semantics for L_2?

Locality principle

Locality principle

Whether or not $Q_M(P)$ is true depends only on M and P and not on what sets and relations exist in general over M.

This means that we are **not forced** to consider full second order logic. Not clear what the argument is for considering Henkin (general) semantics for second order logic.

Alternative proof of the main Theorem?

Suppose Q of type $\langle 1 \rangle$ is implicitly defined by σ.
Fix a universe M and for every $A \subseteq M$ let

$$M_A = (M, \{ A \})$$

be the L_2 model in which the predicate variables range over the singleton set $\{ A \}$.
σ may not include n-ary predicate symbols for $n \geq 2$.
Let $Q'_M = \mathcal{P}(M)$ be the universally true second order predicate.
Then $(M_A, Q'_M) \models \sigma$ iff $Q'_M \cap \{ A \} = Q_M \cap \{ A \}$ iff $A \in Q_M$.
Let φ be the first-order formula we get from σ by removing second-order quantifiers and replacing all predicate variables by the single predicate variable P. Also repacing all $\mathbf{Q}(P)$ by $\top$. Then

$$(M, A) \models \varphi \text{ iff } (M_A, Q'_M) \models \sigma \text{ iff } A \in Q_M$$

and thus φ defines Q.

Why use L_2?

The black (red?) box

The Main Theorem: plugging in L_2 in the machine outputs FOL:

$$\mathrm{Beth}^2(L_2, \mathrm{FOL})$$

Full second order logic is clearly 'to strong:' Q_0 is implicitly definable in second order logic.
What about Π^1_1, or some other fragment more to the nature of deductive rules?

Is this approach really inferential?

- Clearly having a characterization in terms of deduction rules gives implicit definability (in some second order logic).
- What about the other way around?

LUNCH