DEPENDENCE IN LOGIC

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**Variable Dependence and Branching**
Motivation from natural languages

Some relative of each villager and some relative of each townsmen hate each other. (Hintikka 1974)

\[ \forall x \exists y \forall z \exists w (V(x) \land T(z) \rightarrow (R(x, y) \land R(z, w) \land H(y, w))) \]

\[ \left( \forall x \exists y \right) \left( \forall z \exists w \right) (V(x) \land T(z) \rightarrow (R(x, y) \land R(z, w) \land H(y, w))) \]

Most of the dots and most of the stars are all connected by lines. (Barwise 1979)

\[ \left( \frac{Q_1 x}{Q_2 y} \right) L(x, y) \]
Branching

For monotone quantifiers the branching of $Q_1$ and $Q_2$

$$
\left( \begin{array}{c}
Q_1x \\
Q_2y
\end{array} \right) R(x, y)
$$

is interpreted as

$$
\text{Br}(Q_1, Q_2)xy R(x, y),
$$

where $\text{Br}(Q_1, Q_2)$ is the new quantifier:

$$
\{ R \mid \exists A \in Q_1, B \in Q_2, A \times B \subseteq R \}.
$$

Example:

$$
R \in \text{Br}(\forall \exists, \forall \exists)
$$

iff

$$
\exists S_1, S_2 \in \forall \exists \text{ such that } S_1 \times S_2 \subseteq R
$$

iff

$$
\exists f, g : M \to M \text{ such that } \forall x, z R(x, f(x), z, g(z))
$$
**Dependence Logic**

Dependence logic: FOL + \( = (x_1, \ldots, x_{n-1}, x_n) \)

A formula is satisfied (or not) by a **set** of assignments, a **team**.

\[
M \models_X = (x_1, \ldots, x_{n-1}, x_n)
\]

iff

for all \( s, s' \in X \) if \( s(x_i) = s'(x_i) \) for all \( i < n \) then \( s(x_n) = s'(x_n) \).

\[
M \models_X \exists x \varphi
\]

iff

there is \( f : X \rightarrow M \) such that \( M \models_X [f/x] \varphi \),

where \( X[f/x] = \{ s[f(s)/x] | s \in X \} \).
Branching in Dependence Logic

\[ M \models \text{Br}(\forall \exists, \forall \exists)xyzw R(x, y, z, w) \]

iff

\[ M \models \forall x \exists y \forall z \exists w ((z, w) \land R(x, y, z, w)) \]

What about generalized quantifiers?

\[ M \models \text{Br}(Q_1, Q_2)xy R(x, y) \]

iff

\[ M \models Q_1 x Q_2 y ((y) \land R(x, y)) \]

?
GENERALIZED QUANTIFIERS IN DEPENDENCE LOGIC
Lifting Quantifiers

In standard Tarskian semantics a quantifier (on a domain $M$) is a function from sets of assignments to sets of assignments:

$$Q : \mathcal{P}(M^{n+1}) \rightarrow \mathcal{P}(M^n).$$

In team semantics we want to lift this function to a function

$$Q : \mathcal{H}(M^{n+1}) \rightarrow \mathcal{H}(M^n),$$

where $\mathcal{H}(M^n)$ is the set of all teams (i.e., sets) of $n$-ary assignments.

Definition

$$M \models_X Qx \varphi \text{ iff there is } F : X \rightarrow Q \text{ such that } M \models_{X[F/x]} \varphi.$$ 

where $X[F/x] = \{ s[a/x] \mid a \in F(s), s \in X \}.$
Quantifiers and dependence

Proposition

Formulas without dependence atoms maintain their meaning when lifted to team semantics.

We want:

\[ M \models Q_1 x Q_2 y (\equiv (y) \land R(x, y)) \iff M \models Br(Q_1, Q_2) x y R(x, y). \]

However, if \( Q_2 \) contains no singleton sets then

\[ M \not\models Q_1 x Q_2 y (\equiv (y) \land R(x, y)). \]

Thus, we need a new dependence atom!
Multivalued Dependence
A course database

<table>
<thead>
<tr>
<th>Course</th>
<th>Student</th>
<th>Credits</th>
<th>Year</th>
</tr>
</thead>
<tbody>
<tr>
<td>LC1510</td>
<td>Svensson</td>
<td>7.5 hp</td>
<td>2010</td>
</tr>
<tr>
<td>LC1510</td>
<td>Johansson</td>
<td>7.5 hp</td>
<td>2011</td>
</tr>
<tr>
<td>LC1520</td>
<td>Svensson</td>
<td>15 hp</td>
<td>2011</td>
</tr>
<tr>
<td>LC1520</td>
<td>Andersson</td>
<td>15 hp</td>
<td>2011</td>
</tr>
</tbody>
</table>

- \( = \) (Course, Credits)
- \( \text{not} = \) (Course, Student).
- \( F_{\text{Student}} \) takes values for Course and Credits and gives set of possible values for Student.
- \( F_{\text{Student}}(\text{LC1510}, 7.5 \text{ hp}) = \{ \text{Svensson, Johansson} \} \)
- \( F_{\text{Student}} \) is determined by the value of Course.
- \([\text{Course} \rightarrow \text{Student}]\)
- \([\rightarrow] \text{dependent} \) on context.
- \( F_{\text{Student}}(\text{LC1510}, 7.5 \text{ hp}, 2010) = \{ \text{Svensson} \} \)
- \( F_{\text{Student}}(\text{LC1510}, 7.5 \text{ hp}, 2011) = \{ \text{Johansson} \} \)
Multivalued Dependence and Teams

If $s \in X$ then $F^y_X(s) = \{ a \mid s[a/y] \in X \}$.

**Definition**

$M \models_X [\bar{x} \rightarrow y]$ if $F^y_X$ is determined by the values of $\bar{x}$. (Only for $y \notin \bar{x}$.)

**Proposition**

$M \models_X [\bar{x} \rightarrow y]$ iff for all $s, s' \in X$ such that $s(\bar{x}) = s'(\bar{x})$ there exists $s_0 \in X$ such that $s_0(\bar{x}) = s(\bar{x})$, $s_0(y) = s(y)$, and $s_0(\bar{z}) = s'(\bar{z})$, where $\bar{z}$ are the variables in $\text{dom}(X) \setminus (\{ \bar{x} \} \cup \{ y \})$.

- $M \models_X [\bar{x} \rightarrow y]$ is dependent on context and not closed downwards.
- $M \models_X (\bar{x}, y)$ iff $X \models [\bar{x} \rightarrow y]$ and $F^y_X$ only takes singleton values.
GENERALIZED QUANTIFIERS AND MULTIVALUED DEPENDENCE

PROPOSITION

If $Q$ is monotone then $M \models Br(Q, Q)xy R(x, y)$ iff

$$M \models Qx Qy \left( \left[ \rightarrow y \right] \land R(x, y) \right).$$

PROPOSITION

FOL + multivalued dependencies has the same strength, on the level of sentences, as ESO, and thus as Dependence Logic.

PROPOSITION (GALLIANI 2011)

FOL + multivalued dependencies has the same strength as ESO also on the level of open formulas (not true for dependence logic).
Thank you for your attention.