Dependence in logic

Filosofidagarna 2011

Fredrik Engström, Göteborg

June 12, 2011

Variable dependence and branching

Motivation from natural languages

Some relative of each villager and some relative of each townsmen hate each other. (Hintikka 1974)

$$\forall x \exists y \forall z \exists w \big(V(x) \wedge T(z) \to (R(x, y) \wedge R(z, w) \wedge H(y, w))\big)$$

$$\begin{pmatrix} \forall x \exists y \\ \forall z \exists w \end{pmatrix} \big(V(x) \wedge T(z) \to (R(x, y) \wedge R(z, w) \wedge H(y, w))\big)$$

Most of the dots and most of the stars are all connected by lines. (Barwise 1979)

$$\begin{pmatrix} Q_1 x \\ Q_2 y \end{pmatrix} L(x, y)$$

Branching

For monotone quantifiers the branching of Q_1 and Q_2

$$\begin{pmatrix} Q_1 x \\ Q_2 y \end{pmatrix} R(x, y)$$

is interpreted as

$$\mathrm{Br}(Q_1, Q_2)xy\, R(x, y),$$

where $\mathrm{Br}(Q_1, Q_2)$ is the **new** quantifier:

$$\{\, R \mid \exists A \in Q_1, B \in Q_2, A \times B \subseteq R \,\}\,.$$

Example:

$$R \in \mathrm{Br}(\forall\exists, \forall\exists)$$

$$\text{iff}$$

$$\exists\, S_1, S_2 \in \forall\exists \text{ such that } S_1 \times S_2 \subseteq R$$

$$\text{iff}$$

$$\exists f, g : M \to M \text{ such that } \forall x, z\, R\big(x, f(x), z, g(z)\big)$$

DEPENDENCE LOGIC

Dependence logic: FOL + $=(x_1, \ldots, x_{n-1}, x_n)$
A formula is satisfied (or not) by a **set** of assignments, a **team**.

$$M \models_X =(x_1, \ldots, x_{n-1}, x_n)$$

iff

for all $s, s' \in X$ if $s(x_i) = s'(x_i)$ for all $i < n$ then $s(x_n) = s'(x_n)$.

$$M \models_X \exists x \varphi$$

iff

there is $f : X \to M$ such that $M \models_{X[f/x]} \varphi$,

where $X[f/x] = \{ s[f(s)/x] \mid s \in X \}$.

Branching in Dependence Logic

$$M \vDash \mathrm{Br}(\forall\exists, \forall\exists)\, xyzw\, R(x, y, z, w)$$

iff

$$M \vDash \forall x \exists y \forall z \exists w \left(=\!(z, w) \wedge R(x, y, z, w) \right)$$

What about generalized quantifiers?

$$M \vDash \mathrm{Br}(Q_1, Q_2)\, xy\, R(x, y)$$

iff

$$M \vDash Q_1 x\, Q_2 y \left(=\!(y) \wedge R(x, y) \right)$$

?

Generalized quantifiers in Dependence Logic

Lifting Quantifiers

In standard Tarskian semantics a quantifier (on a domain M) is a function from sets of assignments to sets of assignments:

$$Q : \mathcal{P}(M^{n+1}) \to \mathcal{P}(M^n).$$

In team semantics we want to lift this function to a function

$$Q : \mathcal{H}(M^{n+1}) \to \mathcal{H}(M^n),$$

where $\mathcal{H}(M^n)$ is the set of all teams (i.e., sets) of n-ary assignments.

Definition

$M \vDash_X Qx\varphi$ iff there is $F : X \to Q$ such that $M \vDash_{X[F/x]} \varphi$.

where $X[F/x] = \{\, s[a/x] \mid a \in F(s), s \in X \,\}$.

Quantifiers and dependence

Proposition

Formulas without dependence atoms maintain their meaning when lifted to team semantics.

We want:

$$M \models Q_1 x Q_2 y\big(=(y) \land R(x,y)\big) \text{ iff } M \models \mathrm{Br}(Q_1, Q_2)xyR(x,y).$$

However, if Q_2 contains no singleton sets then

$$M \not\models Q_1 x Q_2 y\big(=(y) \land R(x,y)\big).$$

Thus, we need a new dependence atom!

Multivalued Dependence

A course database

Course	Student	Credits	Year
LC1510	Svensson	7.5 hp	2010
LC1510	Johansson	7.5 hp	2011
LC1520	Svensson	15 hp	2011
LC1520	Andersson	15 hp	2011

- $=(\text{Course}, \text{Credits})$
- **not** $=(\text{Course}, \text{Student})$.
- F^{Student} takes values for Course and Credits and gives set of possible values for Student.
- $F^{\text{Student}}(\text{LC1510}, 7.5\text{ hp}) = \{\text{Svensson}, \text{Johansson}\}$
- F^{Student} is determined by the value of Course.
- $[\text{Course} \twoheadrightarrow \text{Student}]$
- $[\twoheadrightarrow]$ **dependent** on context.
- $F^{\text{Student}}(\text{LC1510}, 7.5\text{ hp}, 2010) = \{\text{Svensson}\}$
- $F^{\text{Student}}(\text{LC1510}, 7.5\text{ hp}, 2011) = \{\text{Johansson}\}$

Multivalued dependence and teams

- If $s \in X$ then $F^y_X(s) = \{ a \mid s[a/y] \in X \}$.

Definition

$M \vDash_X [\bar{x} \twoheadrightarrow y]$ if F^y_X is determined by the values of $\bar{x}$. (Only for $y \notin \bar{x}$.)

Proposition

$M \vDash_X [\bar{x} \twoheadrightarrow y]$ iff for all $s, s' \in X$ such that $s(\bar{x}) = s'(\bar{x})$ there exists $s_0 \in X$ such that $s_0(\bar{x}) = s(\bar{x})$, $s_0(y) = s(y)$, and $s_0(\bar{z}) = s'(\bar{z})$, where $\bar{z}$ are the variables in $dom(X) \setminus (\{ \bar{x} \} \cup \{ y \})$.

- $M \vDash_X [\bar{x} \twoheadrightarrow y]$ is **dependent on context** and **not closed downwards**.
- $M \vDash_X =(\bar{x}, y)$ iff $X \vDash [\bar{x} \twoheadrightarrow y]$ and F^y_X only takes singleton values.

Generalized quantifiers and multivalued dependence

Proposition

If Q is monotone then $M \vDash Br(Q,Q)xy\,R(x,y)$ iff

$$M \vDash Qx\,Qy\left([\twoheadrightarrow y] \wedge R(x,y)\right).$$

Proposition

FOL + multivalued dependencies has the same strength, on the level of sentences, as ESO, and thus as Dependence Logic.

Proposition (Galliani 2011)

FOL + multivalued dependencies har the same strength as ESO also on the level of open formulas (not true for dependence logic).

Thank you for your attention.