

# DEPENDENCE IN LOGIC

LOGICAL CONSTANTS WORKSHOP

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# INTRODUCTION

- ▶ Henkin quantifier:  $(\forall x \exists y / \forall z \exists w)$  (1959)
- ▶ Hintikka and Sandu's IF-logic:  $\forall x \exists y \forall z \exists w / x$  (1989)
- ▶ Hodges' semantics for IF-logic: Using **sets of assignments**. (1997)
- ▶ Vännänen's Dependence Logic: Using dependence as **atomic** property:  $=(x, y, z)$  (2007)

$$\forall x \exists y \forall z \exists w (= (z, w) \wedge \dots)$$

# HODGES' SEMANTICS

- ▶  $X$  is a **team**, i.e., a set of assignments.
- ▶  $M \models_X \varphi$ .
- ▶ For first-order  $\varphi$ :  $M \models_X \varphi$  iff for all  $s \in X$ ,  $M \models_s \varphi$ .
- ▶  $M \models_X = (x, y, z)$  iff there is a function  $f: M^2 \rightarrow M$  such that for every  $s \in X$ :  $s(z) = f(s(x), s(y))$ ; or, equivalently:

$$M \models_X = (x, y, z)$$

iff for all  $s, s' \in X$  if  $s(x) = s'(x)$  and  $s(y) = s'(y)$  then  $s(z) = s'(z)$ .

$x$	$y$	$z$
1	11	11
1	12	11
2	11	2
2	13	2

- ▶  $M \not\models_X x = z$
- ▶  $M \not\models_X x \neq z$
- ▶  $M \models_X = (x, z)$
- ▶  $M \not\models_X = (x, y)$

## HODGES' SEMANTICS II

- ▶  $M \models_X \varphi \wedge \psi$  iff  $M \models_X \varphi$  and  $M \models_X \psi$ .
- ▶  $M \models_X \varphi \vee \psi$  iff there are  $Y$  and  $Z$  such that  $M \models_Y \varphi$ ,  $M \models_Z \psi$  and  $X = Y \cup Z$ .
- ▶  $M \models_X \exists x \varphi$  iff there is  $f: X \rightarrow M$  such that  $M \models_{X[f/x]} \varphi$ .
- ▶  $M \models_X \forall x \varphi$  iff  $M \models_{X[M/x]} \varphi$ .

$$X[f/x] = \{ s[f(s)/x] \mid s \in X \}.$$

$$X[M/x] = \{ s[a/x] \mid a \in M, s \in X \}.$$

## BRANCHING IN NATURAL LANGUAGES

*Some relative of each villager and some relative of each townsmen hate each other. (Hintikka 1974)*

$$\left( \begin{array}{l} \forall x \exists y \\ \forall z \exists w \end{array} \right) (V(x) \wedge T(z) \rightarrow (R(x, y) \wedge R(z, w) \wedge H(y, w)))$$

*Most of the dots and most of the stars are all connected by lines. (Barwise 1979)*

*Two examiners marked six scripts. (Davies 1989)*

## BRANCHING AS AN OPERATOR

For monotone quantifiers the branching of  $Q_1$  and  $Q_2$  as in

$$\begin{pmatrix} Q_1 x \\ Q_2 y \end{pmatrix} R(x, y)$$

can be represented by the quantifier  $\text{Br}(Q_1, Q_2)$  as in  $\text{Br}(Q_1, Q_2)xy R(x, y)$ , where

$\text{Br}(Q_1, Q_2)$  is the quantifier

$$\{ R \mid \exists A \in Q_1, B \in Q_2, A \times B \subseteq R \}.$$

## BRANCHING IN DEPENDENCE LOGIC

$$M \models \text{Br}(\forall\exists, \forall\exists)xyzw R(x, y, z, w)$$

iff

$$M \models \forall x \exists y \forall z \exists w (=(z, w) \wedge R(x, y, z, w))$$

What about generalized quantifiers?

$$M \models \text{Br}(Q_1, Q_2)xy R(x, y)$$

iff

$$M \models Q_1 x Q_2 y (=(y) \wedge R(x, y))$$



# GENERALIZED QUANTIFIERS IN DEPENDENCE LOGIC

# LIFTING FUNCTIONS

The **Hodges space** of order ideals on the power set is

$$\mathcal{H}(A) = \mathcal{L}(\mathcal{P}(A)).$$

Given  $h : \mathcal{P}(A) \rightarrow \mathcal{P}(B)$  we define the **lift**:

$$\mathcal{L}(h) : \mathcal{H}(A) \rightarrow \mathcal{H}(B), \mathcal{X} \mapsto \downarrow \{ h(X) \mid X \in \mathcal{X} \},$$

where  $\downarrow \mathcal{X}$  is the downward closure of  $\mathcal{X}$ , i.e.

$$\downarrow \mathcal{X} = \{ X \mid \exists Y \in \mathcal{X}, X \subseteq Y \}.$$

## LIFTING QUANTIFIERS

- ▶  $Q$  a monotone type  $\langle 1 \rangle$  quantifier.
- ▶  $Q : \mathcal{P}(M^{n+1}) \rightarrow \mathcal{P}(M^n)$
- ▶  $\mathcal{L}(Q) : \mathcal{H}(M^{n+1}) \rightarrow \mathcal{H}(M^n)$
- ▶ Gives truth conditions for  $Q$  in Hodges semantics:

$M \models_X Qx\varphi$  iff there is  $F : X \rightarrow Q$  such that  $M \models_{X[F/x]} \varphi$ .

where  $X[F/x] = \{ s[a/x] \mid a \in F(s) \}$ .

- ▶  $\mathcal{L}(\exists)$  and  $\mathcal{L}(\forall)$  give the same truth conditions for  $\exists$  and  $\forall$  as we had before.

### PROPOSITION

For formulas  $\varphi$  with  $Q$ , but without dependence atoms:

$M \models_X \varphi$  iff for all  $s \in X$ ,  $M \models_s \varphi$ .

# QUANTIFIERS AND DEPENDENCE

If  $Q$  contains no singletons then  $M \not\models_X Qx (= (x) \wedge \varphi)$ .

$$M \models \text{Br}(Q_1, Q_2)xy R(x, y)$$

iff

$$M \models Q_1x Q_2y (= (y) \wedge R(x, y))$$

# MULTIVALUED DEPENDENCE

## A COURSE DATABASE

Course	Student	Credits	Year
LC1510	Svensson	7.5 hp	2010
LC1510	Johansson	7.5 hp	2011
LC1520	Svensson	15 hp	2011
LC1520	AnderssonJohansson	15 hp	2011

- ▶  $=(\text{Course}, \text{Credits})$
- ▶ It is not true that  $=(\text{Course}, \text{Student})$ .
- ▶  $F^{\text{Student}}$  takes values for Course and Credits and gives a set of possible values for Student.
- ▶  $F^{\text{Student}}(\text{LC1510}, 7.5 \text{ hp}) = \{ \text{Svensson}, \text{Johansson} \}$
- ▶  $F^{\text{Student}}$  is determined by the value of Course.
- ▶  $[\text{Course} \rightarrow \text{Student}]$
- ▶  $[\rightarrow]$  **dependent** on context.
- ▶  $F^{\text{Student}}(\text{LC1510}, 7.5 \text{ hp}, 2010) = \{ \text{Svensson} \}$
- ▶  $F^{\text{Student}}(\text{LC1510}, 7.5 \text{ hp}, 2011) = \{ \text{Johansson} \}$
- ▶  $[\rightarrow]$  **not** closed downwards: **Not** true that  $[\rightarrow \text{Student}]$

# MULTIVALUED DEPENDENCE AND TEAMS

- ▶ If  $s \in X$  then  $F_X^y(s) = \{ a \mid s[a/y] \in X \}$ .

## DEFINITION

$M \models_X [\bar{x} \rightarrow y]$  if  $F_X^y$  is determined by the values of  $\bar{x}$ . (Only for  $y \notin \bar{x}$ .)

## PROPOSITION

$M \models_X [\bar{x} \rightarrow y]$  iff for all  $s, s' \in X$  such that  $s(\bar{x}) = s'(\bar{x})$  there exists  $s_0 \in X$  such that  $s_0(\bar{x}) = s(\bar{x})$ ,  $s_0(y) = s(y)$ , and  $s_0(\bar{z}) = s'(\bar{z})$ , where  $\bar{z}$  are the variables in  $\text{dom}(X) \setminus (\{ \bar{x} \} \cup \{ y \})$ .

- ▶  $M \models_X [\bar{x} \rightarrow y]$  is **dependent on context** and **not closed downwards**.
- ▶  $M \models_X = (\bar{x}, y)$  iff  $X \models [\bar{x} \rightarrow y]$  and  $F_X^y$  only takes singleton values.

## GENERALIZED QUANTIFIERS AND MULTIVALUED DEPENDENCE

## PROPOSITION

If  $Q$  is monotone then  $M \models \text{Br}(Q_1, Q_2)xyR(x, y)$  iff

$$M \models Q_1 x Q_2 y ([\rightarrow y] \wedge R(x, y)).$$

## PROPOSITION

FOL + multivalued dependencies has the same strength, on the level of sentences, as ESO, and thus as Dependence Logic.

## PROPOSITION [GALLIANI -11]

The class of teams definable in FOL + multivalued dependencies are exactly the ones definable in ESO (with an extra predicate for the team).

# EMBEDDED MULTIVALUED DEPENDENCE

- ▶ Multivalued dependence is axiomatizable (as an atomic property).
- ▶ Multivalued dependence is **dependent on context**.

## DEFINITION

$M \models_X [\bar{x} \rightarrow \bar{y} | \bar{z}]$  iff  $Y \models [\bar{x} \rightarrow \bar{y}]$  where  $Y$  is the projection of  $X$  onto  $\{\bar{x}, \bar{y}, \bar{z}\}$ .

- ▶  $[\bar{x} \rightarrow \bar{y} | \bar{z}]$  is **independent on context**.
- ▶ This is the **independence atom** introduced by Väänänen and Grädel:  $\bar{y} \perp_{\bar{x}} \bar{z}$  iff  $[\bar{x} \rightarrow \bar{y} | \bar{z}]$
- ▶ However, embedded multivalued dependence is **not** axiomatizable. [Sagiv Walecka 1982] (Which both functional dependence and multivalued dependence are.)
- ▶ Embedded multivalued dependence is definable in FOL with multivalued dependencies. [Galliani -11]

THANK YOU FOR YOUR  
ATTENTION.