Dependence Logic and Generalized Quantifiers

Logics for Dependence and Independence, Dagstuhl

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Fredrik Engström. Generalized quantifiers in dependence logic. **Journal of Logic, Language and Information**, 21:299–324, 2012. ISSN 0925-8531.

Fredrik Engström and Juha Kontinen. Characterizing quantifier extensions of dependence logic. **Journal of Symbolic Logic**, 78(1):307–316, 2013.

Fredrik Engström, Juha Kontinen, and Jouko Väänänen. Dependence logic with generalized quantifiers: Axiomatizations. **Journal of Computer and System Sciences**, to appear.

Some relative of each villager and some relative of each townsmen hate each other.

$$\begin{pmatrix} \forall x \, \exists y \\ \forall z \, \exists w \end{pmatrix} A(x, y, z, w)$$

Most philosophers and most linguists agree with each other about branching quantification.

$$\begin{pmatrix} Q_1 x \\ Q_2 y \end{pmatrix} A(x, y)$$

Generalized quantifiers

A generalized quantifier Q is a class of structures closed under isomorphism in a fixed signature.

- $Q_M = \{ R \mid (M, R) \in Q \}$.

$$Q_M \subseteq \mathcal{P}(M).$$

$$M, s \models Qx\,\phi \text{ iff } \phi^{M,s} \in Q_M$$

- $\forall_M = \{ M \}$
- $\exists_M = \{ A \subseteq M \mid A \neq \emptyset \}$
- $(Q_0)_M = \{ A \subseteq M \mid |A| \geq \aleph_0 \}$
- $(Q_1)_M = \{ A \subseteq M \mid |A| \geq \aleph_1 \}$
- $(Q_R)_M = \{ A \subseteq M \mid |A| > |M \setminus A| \}$

Q is **monotone increasing** if $A \subseteq B$ and $A \in Q_M$ implies $B \in Q_M$.

Branching

$$\begin{pmatrix} Q_1 x \\ Q_2 y \end{pmatrix} \phi$$

For monotone increasing quantifiers:

$$\mathrm{Br}(Q_1, Q_2)_M = \left\{ R \subseteq M^2 \;\middle|\; A \times B \subseteq R, A \in (Q_1)_M, B \in (Q_2)_M \right\}$$

$$\mathrm{Br}(Q_1, Q_2) xy \phi \equiv \begin{pmatrix} Q_1 x \\ Q_2 y \end{pmatrix} \phi$$

Iteration

$$(Q_1 \cdot Q_2)_M = \left\{ R \subseteq M^2 \;\middle|\; \{ a \mid {}_aR \in (Q_2)_M \} \in (Q_1)_M \right\}$$

$$(Q_1 \cdot Q_2) xy \phi \equiv Q_1 x \, Q_2 y \, \phi$$

Dependence Logic with Q

Only monotone increasing unary quantifiers.

- $D(Q)$ is $\phi ::= \gamma \mid \phi \wedge \phi \mid \phi \vee \phi \mid \exists x \phi \mid \forall x \phi \mid Qx \phi$, where γ is a literal or dependence atom.
- $M \vDash \sigma$ iff $M, \{\emptyset\} \vDash \sigma$.

- $M, X \vDash \gamma$ if for all $s \in X$: $M, s \vDash \gamma$, where γ is a literal.
- $M, X \vDash =(\bar{t}, t')$ if for all $s, s' \in X$ if $s(\bar{t}) = s'(\bar{t})$ then $s(t') = s'(t')$.
- $M, X \vDash \phi \wedge \psi$ if $M, X \vDash \phi$ and $M, X \vDash \psi$.
- $M, X \vDash \phi \vee \psi$ if there are $Y \cup Z = X$ such that $M, Y \vDash \phi$ and $M, Z \vDash \psi$.

- $M, X \vDash \exists x \phi$ if there is $f : X \to M$ s.t. $M, X[f/x] \vDash \phi$
- $M, X \vDash \forall x \phi$ if $M, X[M/x] \vDash \phi$

$X[f/x] = \{ s[f(s)/x] \mid s \in X \}$ and $X[M/x] = \{ s[a/x] \mid s \in X, a \in M \}$

$M, X \vDash Qx\phi$?

Conservative over $FO(Q)$

$$M, X \vDash \phi \text{ iff for all } s \in X, M, s \vDash \phi$$

for all FO(Q)-formulas ϕ.

Respect the quantifiers
The truth conditions of $\exists$ and $\forall$ should be special cases of the general condition.

Respect iteration

$$M, X \vDash (Q_1 \cdot Q_2)xy\phi \text{ iff } M, X \vDash Q_1 x Q_2 x \phi$$

Express branching
Be able to express

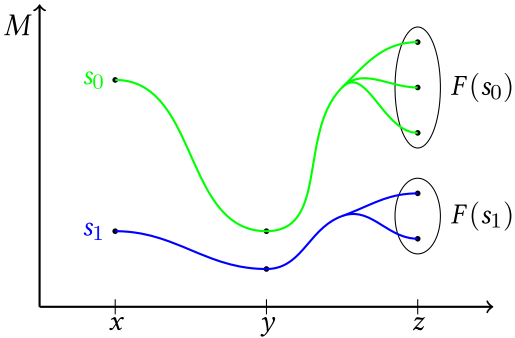
$$\mathrm{Br}(Q_1, Q_2)xy\phi.$$

Quantifiers in dependence logic

- $M, X \vDash Qx\,\phi$ iff there is $F : X \to Q_M$ such that $M, X[F/x] \vDash \phi$.

$$X[F/x] = \{\, s[a/x] \mid s \in X, a \in F(s) \,\}$$

Example: $M, \{s_0, s_1\} \vDash \exists^{\geq 2} z\, Rxyz$



Properties of Dependence logic

- $M, \emptyset \vDash \phi$
- **Downwards closure:** If $Y \subseteq X$ and $M, X \vDash \phi$ then $M, Y \vDash \phi$.
- Branching of generalized quantifiers is expressible in $D(Q)$.

$$\mathrm{Br}(Q, Q)xy\,\phi(x, y, \bar{z}) \equiv$$

$$\begin{aligned}
\exists w, w' \big(\quad & =(\bar{z}, w) \wedge =(\bar{z}, w') \wedge \\
& Qx\,\exists y\,(y = w \wedge =(\bar{z}, x, y) \wedge \\
& Qx'\,\exists y'\,(y' = w' \wedge =(\bar{z}, x', y') \wedge \\
& \forall u\,\exists v\,(=(\bar{z}, u, v) \wedge (x = u \to v = w) \wedge \\
& \forall u'\,\exists v'\,(=(\bar{z}, u', v') \wedge (x' = u' \to v' = w') \wedge \\
& ((v = w \wedge v' = w') \to \phi(u, u', \bar{z}))))) \big)
\end{aligned}$$

$$\equiv Qx\,Qy\,(x \perp_{\bar{z}} y \wedge \phi(x, y, \bar{z}))$$

Strength

Theorem

$$D(Q) \equiv \mathrm{ESO}(Q)$$

Theorem

Every $D(Q)$ formula is equivalent to one of the form:

$$\mathcal{H}^1 x_1 \ldots \mathcal{H}^m x_m \exists y_1 \ldots \exists y_n \Big(\bigwedge_{1 \leq i \leq n} =(\overline{x}^i, y_i) \wedge \theta \Big),$$

where $\mathcal{H}^i$ is either Q or $\forall$, and θ is a quantifier-free FO formula.

Axiomatization

- $\breve{Q}$ is the dual of Q: "$\breve{Q} = \neg Q \neg$"

Axiomatize $\mathrm{FO}(Q, \breve{Q})$ consequences.

Idea:

- Construct a natural deduction system in which the normal form can be derived.
- Allow dependencies in normal forms to be replaced by **finite approximations**.
- Show that in enough models (recursively saturated) the set of finite approximations is equivalent to the original sentence.

Axiomatizing $D(Q, \breve{Q})$ I: General rules

- Standard rules for $\mathrm{FO}(Q, \breve{Q})$ formulas.
- Standard rules for conjunction, existential quantifier, and universal quantifier.
- Commutativity, associativity and monotonicity of disjunction.
- Monotonicity, extending scope, and renaming of bound variables for Q and $\breve{Q}$.
- Duality of $\breve{Q}$ with respect to $\mathrm{FO}(Q, \breve{Q})$ formulas.

Axiomatizing $D(Q, \check{Q})$ II: Dependence related rules

- Unnesting:

$$\frac{=(t_1, ..., t_n)}{\exists z(=(t_1, ..., z, ..., t_n) \wedge z = t_i)}$$

where z is a new variable.

- Dependence distribution:

$$\frac{\exists y_1 \dots \exists y_n(\bigwedge_{1 \leq j \leq n} =(\bar{z}^j, y_j) \wedge \phi) \vee \exists y_{n+1} \dots \exists y_m(\bigwedge_{n+1 \leq j \leq m} =(\bar{z}^j, y_j) \wedge \psi)}{\exists y_1 \dots \exists y_m(\bigwedge_{1 \leq j \leq m} =(\bar{z}^j, y_j) \wedge (\phi \vee \psi))}$$

where ϕ and ψ are quantifier free FO formulas.

- Dependence introduction:

$$\frac{\exists x \mathcal{H} y \phi}{\mathcal{H} y \exists x (=(\bar{z}, x) \wedge \phi)}$$

where $\bar{z}$ lists the variables in $\mathrm{FV}(\phi) - \{x, y\}$ and $\mathcal{H} \in \{\forall, Q, \check{Q}\}$.

Approximations

Suppose σ is in normal form:

$$\mathcal{H}^1 x_1 \dots \mathcal{H}^m x_m \exists y_1 \dots \exists y_n \Big(\bigwedge_{1 \le i \le n} =(\bar{x}^i, y_i) \wedge \theta(\bar{x}, \bar{y}) \Big).$$

Let $A^k\sigma$ be

$$\forall \bar{x}_1 \exists \bar{y}_1 \dots \forall \bar{x}_k \exists \bar{y}_k \Big(\bigwedge_{1 \le j \le k} R(\bar{x}_j) \to \bigwedge_{1 \le j \le k} \theta(\bar{x}_j, \bar{y}_j) \wedge \bigwedge_{\substack{1 \le i \le n \\ 1 \le j, j' \le k}} (\bar{x}^i_j = \bar{x}^i_{j'} \to y_{i,j} = y_{i,j'}) \Big)$$

Let $B\sigma$ be

$$\mathcal{H}^1 x_1 \dots \mathcal{H}^m x_m R(x_1, \dots, x_m).$$

Axiomatizing $D(Q, \breve{Q})$ III: The approximation rule

$$\dfrac{\sigma \qquad \begin{matrix} [B\sigma] & & [A^k\sigma] \\ \ddots & & \iddots \\ & \psi & \end{matrix}}{\psi}\ (\text{Approx})$$

where σ is a sentence in normal form, and R does not appear in ψ nor in any uncancelled assumptions in the derivation of ψ, except for $B\sigma$ and $A^k\sigma$.

Completeness for weak semantics

Let $\Gamma \vDash_w \phi$ mean that $\Gamma \vDash \phi$ for any monotone increasing (non-trivial) interpretation of Q (and $\check{Q}$ is interpreted as the dual of the interpretation of Q).

Theorem
This system is sound and complete wrt $\Gamma \vDash_w \phi$ where ϕ is $\mathrm{FO}(Q, \check{Q})$.

Non-montone quantifiers

$\mathrm{Br}(Q_1, Q_2)$ may be defined for a rather wide range of quantifiers.

$$M \models \exists^{<5} x\, Px$$

$$M \models \exists^{=5} x\, Px$$

a formula ϕ is satisfied by a team X if for every assignment $s : \mathrm{dom}(X) \to M^k$, if $s \in X$ then s satisfies ϕ.

a formula ϕ is satisfied by a team X if for every assignment $s : \mathrm{dom}(X) \to M^k$, $s \in X$ iff s satisfies ϕ.

I.e., $M, X \models \phi$ iff $X = \phi(M)$ (for first-order ϕ).

Maximal semantics

- $M, X \models_m \psi$ if $M, X \models \psi$ and for all $Y \supsetneq X : M, Y \not\models \psi$, for literals ψ.
- $M, X \models_m \phi \wedge \psi$ if $\exists Y, Z$ s.t. $X = Y \cap Z$, and both $M, Y \models \phi$ and $M, Z \models \psi$
- $M, X \models_m \phi \vee \psi$ if $\exists Y, Z$ s.t. $X = Y \cup Z$, and both $M, Y \models \phi$ and $M, Z \models \psi$
- $M, X \models_m Qx \phi$ if $\exists Y$ s.t. $Qx\, Y = X$ and $M, Y \models \phi$

Conclusion

Extending dependence logic with generalized quantifiers is a natural and stable extension.

- $D(Q)$ properly extends both $\mathrm{FO}(Q)$ and D.
- $D(Q)$ is equivalent to $\mathrm{ESO}(Q)$.
- $D(Q)$ has a prenex normal form theorem.
- Similar completeness results as for D.

What about non-monotonic quantifiers?

That's all folks!